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An elliptic Neumann problem with critical nonlinearity in three dimensional domains

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Résumé. On étudie une équation aux dérivées partielles elliptiques du second ordre avec conditions au bord de Neumann. Une analyse asymptotique améliorée permet d'établir en dimension 3 des résultats précédemment connus pour des dimensions supérieures : existence et multiplicité de solutions concentrées, en fonction des propriétés de la courbure moyenne de la frontière du domaine.

Abstract. We study an elliptic partial differential equation of second order with critical nonlinearity and Neumann boundary conditions. An improved asymptotic analysis allows us to prove in dimension 3 results previously known in larger dimensional spaces, i.e. existence and multiplicity of concentrated solutions in connection with properties of the mean curvature of the domain.

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1 Introduction and Results

How strongly differentiated structures may spontaneously emerge from an homogeneous medium ? A mechanism of pattern formation in some biological phenomena has been identified, which relies on activator-inhibitor systems involving self-enhancement (autocatalysis) and long range inhibition [19] (see also [15]). The evolution of activator and inhibitor concentrations is ruled by two coupled parabolic partial differential equations, leading to steady states which depend on a unique elliptic problem, writing as :

$$\begin{cases} -\Delta u + \mu u &= u^p, u > 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega\ . \end{cases} \tag{1.1}$$

$\mu > 0$ and $p > 1$ are constants involving the significant parameters as catalysis and diffusion rates. In the following, we assume that Ω is a smooth and bounded domain in $\mathbb{R}^n$ ($n = 3$, possibly 2, are the more interesting cases in view of modelling). Setting $\tilde{u}(x) = \mu^{\frac{1}{1-p}} u\left(\frac{x - x_\mu}{\sqrt{\mu}}\right)$, (1.1) is equivalent to

$$\begin{cases} -\Delta \tilde{u} + \tilde{u} &= \tilde{u}^p, \tilde{u} > 0 \quad \text{in } \Omega_\mu \\ \frac{\partial \tilde{u}}{\partial \nu} &= 0 \quad \text{on } \partial\Omega_\mu \end{cases} \tag{1.2}$$

with $\Omega_\mu = \sqrt{\mu}\Omega + x_\mu$. Consequently, the results that we shall establish concerning (1.1), which hold provided μ is large enough, apply to (1.2) on sufficiently large domains.

In the subcritical case, i.e. $n = 1, 2$ or $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, we know that (1.1) has for small μ a unique solution which is the constant $\mu^{\frac{1}{p-1}}$, whereas there exists, for μ large enough, at least one nonconstant solution which concentrates at one point of the boundary as μ goes to infinity [16] [17]. In the critical case, i.e. $p = \frac{n+2}{n-2}$, studying (1.1) is more difficult since, from a variational point of view, the problem is noncompact. (Actually, the results may be different : when Ω is a ball and $4 \leq n \leq 6$, (1.1) has a nonconstant solution for any $\mu > 0$ [6] [10].) On $H^1(\Omega)$ we define the functional

$$I_\mu(u) = \frac{\int_\Omega |\nabla u|^2 + \mu \int_\Omega u^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}} \tag{1.3}$$

whose critical points u such that $u \not\equiv 0$ and u has constant sign are, up to a multiplicative constant, solutions of (1.1). I_μ does not satisfy the Palais-Smale condition : there exist sequences of $H^1(\Omega)$ along which I_μ is bounded, I'_μ goes to zero in $H^1(\Omega)$, and which do not contain any convergent subsequence in $H^1(\Omega)$. The concentration-compactness principle [18] provides us with a precise description of such sequences, from which it appears that I_μ, globally noncompact, satisfies the Palais-Smale condition under the level $S/2^{2/n}$, S denoting the Sobolev embedding constant

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ u \not\equiv 0}} \frac{|u|^2_{H_0^1}}{|u|^2_{p+1}} = n(n-2)\left(\frac{\pi^{\frac{n+1}{2}}}{2^{n-1}\Gamma(\frac{n+1}{2})}\right)^{\frac{2}{n}} . \tag{1.4}$$

Using suitably chosen test-functions, Adimurthi and Mancini proved that the infimum of I_μ in $H^1(\Omega)$ is strictly less than $S/2^{2/n}$ [1]. As, on the other hand, the energy of the constant solution, $I_\mu(u \equiv \mu^{\frac{1}{p-1}}) = \mu|\Omega|^{\frac{p-1}{p+1}}$, goes to infinity as μ goes to infinity, (1.1) has at least one nontrivial solution for μ large enough (see also [25]). Since this first result, many works have refined our knowledge of the feature in the critical case. Adimurthi, Pacella and Yadava showed that for μ large enough, a solution u_μ of (1.1) with low energy, i.e. $I_\mu(u_\mu) < S/2^{2/n}$, achieves its supremum at a unique point y_μ which lies on the boundary of the domain [4] (see also [20]). Up to a subsequence, y_μ converges to some limit $y^0 \in \partial\Omega$ as μ goes to infinity, and along this subsequence

$$|\nabla u_\mu|^2 \rightharpoonup \frac{S^{n/2}}{2}\delta_{y^0} \text{ as } \mu \to +\infty \tag{1.5}$$

in the sense of measures, δ_{y^0} denoting the Dirac mass at y^0. It was also proved, in the case $n \geq 7$ and u_μ minimizes I_μ, that the sequence (y_μ) maximizes the mean curvature H of the boundary. If u_μ is not a minimizer of I_μ, the accumulation points of (y_μ) are however critical points of H [5], result extended to the case $n \geq 5$ by Z.Q.Wang [26].

Conversely, assuming that $n \geq 7$ [4], or only $n \geq 5$ [26], we know that for any $y^0 \in \partial\Omega$ a strict local minimizer of H, with $H(y^0) > 0$, there exists for large μ a low energy solution of (1.1) which concentrates at y^0 as μ goes to infinity. The same result holds, assuming that $n \geq 6$, for y^0 a nondegenerate critical point of H, with $H(y^0) > 0$ [3] [23]. Moreover, if $n \geq 5$ and $a > 0$ is an isolated critical value of H which induces a difference of topology between the level sets of H (the relative homology H_* $(H^{a+\delta}, H^{a-\delta})$ is nontrivial for $\delta > 0$ small enough), there exists a solution u_μ of (1.1) which, up to a subsequence, concentrates at a point $y^0 \in \partial\Omega$ such that $H(y^0) = a$ as μ goes to infinity [23].

Unfortunately, the arguments which settle these results do not allow to conclude in low dimensional spaces, whereas the three dimensional case seems of major interest. The same kind of behaviour is expected in any dimension, but some cautiousness is to be recommended, as it is shown by the qualitative difference which occurs between the cases $n = 3$ and $n \geq 4$, concerning a similar problem with Dirichlet boundary conditions [9]. Unless on special domains with symmetries [28] [29], very few is known which holds for any $n \geq 3$, excepted the existence for large μ of a nontrivial low energy solution, and some multiplicity results of such solutions related to geometric properties of the boundary. Actually, Adimurthi and Mancini proved that (1.1) has at least $cat(H^+, \partial\Omega)$ nontrivial solutions for μ large enough, with $H^+ = \{y \in \partial\Omega / H(y) > 0\}$ [2] (see also [27]).

The aim of this paper is to ameliorate our understending of the low energy solutions, in such a way that we may extend to every dimensions $n \geq 3$ results previously known for $n \geq 5, 6$ or 7, and improve some other results as multiplicity of solutions in connection with topological and geometrical properties of the boundary. For sake of simplicity, we shall concentrate our attention on the most difficult case, that is $n = 3$, and we denote by (P) the corresponding problem

$$(P) \begin{cases} -\Delta u + \mu u & = u^5, u > 0 \quad \text{in } \Omega \subset \mathbb{R}^3 \\ \frac{\partial u}{\partial \nu} & = 0 \quad\quad\quad\quad\quad \text{on } \partial\Omega \ . \end{cases}$$

We prove :

Theorem 1.1 $n = 3$. *Let u_μ be a minimizer of I_μ and, for μ large enough, $y_\mu \in \partial\Omega$ the unique point at which u_μ achieves its maximum in $\bar{\Omega}$. Then, $H(y_\mu)$ goes to $\bar{H} = \max_{y \in \partial\Omega} H(y)$ as μ goes to infinity.*

Some additional informations about u_μ may be obtained. Namely :

$$(1.6) \qquad \mathrm{Log}\left(\frac{3\pi}{8} - \frac{1}{2\pi^{1/3}} I_\mu(u_\mu)\right) = -\frac{2\sqrt{\mu}}{\bar{H}} - \mathrm{Log}\frac{2\sqrt{\mu}}{\bar{H}} - \frac{1}{2} - \gamma + O\left(\frac{1}{\sqrt{\mu}}\right)$$

γ denoting the Euler constant, and

$$(1.7) \qquad \mathrm{Log}|u_\mu|_\infty \sim \frac{\sqrt{\mu}}{\bar{H}} \quad \text{as } \mu \to +\infty \ .$$

Moreover, Cherrier [11] proved that for every $\eta > 0$, there exists a constant $C(\eta) > 0$, $C(\eta) = O(\frac{1}{\eta})$ as η goes to zero, such that

$$(1.8) \qquad \left(\frac{S}{2^{2/n}} - \eta\right)\left(\int_\Omega |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \int_\Omega |\nabla u|^2 + C(\eta) \int_\Omega u^2 \qquad \forall u \in H^1(\Omega) \ .$$

Actually, if $n \geq 7$, the best constant $C(\eta)$ satisfies

$$C(\eta) = \frac{C_n \bar{H}^2}{\eta} + o\left(\frac{1}{\eta}\right) \quad \text{as } \eta \to 0 \ .$$

with $C_n > 0$ a constant which depends on n only [5]. If $n = 3$, $S = \frac{3\pi^{4/3}}{2^{4/3}}$, $\frac{S}{2^{2/n}} = \frac{3\pi^{4/3}}{4}$, and we show :

Proposition 1.1 *If $n = 3$, the best constant $C(\eta)$ in (1.8) satisfies*

$$C(\eta) = \frac{\bar{H}^2}{4}\left(\mathrm{Log}\eta + \mathrm{Log}|\mathrm{Log}\eta| + \frac{1}{2} + \gamma - \mathrm{Log}2\pi^{1/3} + O(\frac{1}{|\mathrm{Log}\eta|})\right)^2 \tag{1.9}$$

as $\eta \to 0$.

Conversely to Theorem 1.1, we prove :

Theorem 1.2 $n = 3$. *Let $y_0 \in \partial\Omega$ be a strict local maximum point of the mean curvature H of the domain boundary. For μ large enough, there exists a low energy solution of (P), which concentrates at y_0 as μ goes to infinity.*

Furthermore, denoting by f^c and f_c respectively the lower and upper level sets of a function f, i.e.

$$f^c = \{x \in \mathrm{Dom}(f)/f(x) < c\} \qquad f_c = \{x \in \mathrm{Dom}(f)/f(x) > c\}$$

we state :

Theorem 1.3 *Let $a > 0$ be an isolated critical value of the mean curvature H of $\partial\Omega$, such that the relative homology $H_*(H_{a-\delta}, H_{a+\delta})$ is nontrivial, for any $\delta > 0$ sufficiently small. There exists, for μ large enough, a low energy solution u_μ of (P) which, up to a subsequence, concentrates at a point $y_0 \in \partial\Omega$ such that $H(y_0) = a$, as μ goes to infinity.*

This statement extends to the case $n = 3$ a result previously known for $n \geq 5$ [23]. (We should also have $H'(y_0) = 0$, as it is proved for $n \geq 6$.) Note that the solution u_μ that we find is such that

$$\mathrm{Log}(\frac{3\pi}{8} - \frac{1}{2\pi^{1/3}}I_\mu(u_\mu)) \sim -\frac{2\sqrt{\mu}}{a} \quad \text{as } \mu \to +\infty \tag{1.10}$$

and

$$\mathrm{Log}|u_\mu|_\infty \sim \frac{\sqrt{\mu}}{a} \quad \text{as } \mu \to +\infty\ . \tag{1.11}$$

The same kind of arguments provides us with :

Theorem 1.4 *Let $c > 0$. For μ large enough, (P) has at least as many nontrivial solutions as $\mathrm{cat}(H_{c_\mu}, H_c)$, with $c_\mu = c(1 + \frac{\rho_c}{\mu})$, and ρ_c is some strictly positive constant which depends on c only.*

As a consequence, we recover the multiplicity result of [27] :

Corollary 1.1 *For μ large enough, (P) has at least as many solutions as $\sup_{c>0}\mathrm{cat}(H_c, \partial\Omega)$.*

Indeed, $cat(H_{c_\mu}, H_c) \geq cat(H_{c_\mu}, \partial\Omega)$, and $H_{c_\mu} \supset H_{2c}$ for μ large enough. Then, for μ large enough, (P) has at least as many solutions as $cat(H_{2c}, \partial\Omega)$. $cat(H_{2c}, \partial\Omega)$ is an integer which is less than $cat(\partial\Omega, \partial\Omega)$, and $cat(\partial\Omega, \partial\Omega)$ is finite since $\partial\Omega$ is a compact manifold. Therefore, there exists $c_0 > 0$ such that $\sup_{c>0} cat(H_c, \partial\Omega) = cat(H_{2c_0}, \partial\Omega)$, hence the result. From Theorem 1.4, we also derive:

Corollary 1.2 *Assume that* 0 *is a regular value, or an isolated critical value of* H. *For* μ *large enough, (P) has at least as many nontrivial solutions as* $\mathrm{cat}(H^+, H^+)$, *with* $H^+ = H_0 = \{y \in \partial\Omega / H(y) > 0\}$.

Indeed, there exists $\delta > 0$ such that H has no critical value in $(0, \delta)$. Then, $cat(H_{c_\mu}, H_c) = cat(H^+, H^+)$ as soon as $c_\mu < \delta$. Choosing c such that $0 < c < \delta$, $c_\mu < \delta$ for μ large enough and Corollary 1.2 follows.

The proof of the theorems relies on a precised asymptotic analysis of the functional I_μ in a neighbourhood of the possible low energy solutions of (P). We recall that the solutions of the problem

$$-\Delta u = u^{\frac{n+2}{n-2}}, u > 0 \text{ in } \mathbb{R}^n, n \geq 3 \tag{1.12}$$

such that $u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, $\nabla u \in L^2(\mathbb{R}^n)$, write as

$$U_{\lambda,y}(x) = \frac{\lambda^{\frac{n-2}{2}}}{(1 + \lambda^2|x-y|^2)^{\frac{n-2}{2}}} \qquad \lambda \in \mathbb{R}_+^* \qquad y \in \mathbb{R}^n \tag{1.13}$$

up to the multiplicative constant $\bar{\alpha} = (n(n-2))^{\frac{n-2}{4}}$. In the critical case $p = \frac{n+2}{n-2}$, a blow up technique [14] shows that the low energy solutions of (1.1) are perturbations of the functions $U_{\lambda,y}$, with $y \in \partial\Omega$ and λ goes to infinity as μ goes to infinity [4]. Conversely, one may try to construct low energy solutions of (1.1) in a neighbourhood of the approximate solutions that the $U_{\lambda,y}$'s are. Unfortunately, in low dimensional spaces these functions are not good enough approximations of solutions to make a perturbation approach effective. We have

$$-\Delta\bar{\alpha}U_{\lambda,y} + \mu\bar{\alpha}U_{\lambda,y} - (\bar{\alpha}U_{\lambda,y})^p = \mu\bar{\alpha}U_{\lambda,y} \; . \tag{1.14}$$

In order to get better approximate solutions to the problem, we could substract to $\bar{\alpha}U_{\lambda,y}$ a function ψ which satisfies $-\Delta\psi + \mu\psi = \mu\bar{\alpha}U_{\lambda,y}$ in Ω, defined as

$$\psi(x) = \mu\bar{\alpha}\int_\Omega G_\mu(x,z)U_{\lambda,y}(z)dz$$

G_μ denoting the Green's function of the operator $-\Delta + \mu$ on Ω with Neumann boundary conditions. However, in view of further computations it is preferable to substitute to the integral definition of ψ an approximate function φ given explicitely. In $\mathbb{R}^n$, there exists a unique tempered distribution E such that $-\Delta E + \mu E = \delta_0$ [13], which is

$$E(x) = \frac{1}{\pi}(\frac{\sqrt{\mu}}{2\pi|x|})^{\frac{n-2}{2}} K_{\frac{n-2}{2}}(\sqrt{\mu}|x|) \tag{1.15}$$

where, according to standard notations [24], K_α is a Bessel function

$$(1.16)\qquad K_\alpha(t) = \frac{\pi}{2\sin\alpha\pi}\left(e^{i\alpha\frac{\pi}{2}}J_{-\alpha}(it) - e^{-i\alpha\frac{\pi}{2}}J_\alpha(it)\right) \quad \alpha \notin \mathbb{N}$$

with

$$(1.17)\qquad J_\alpha(t) = \left(\frac{t}{2}\right)^\alpha \sum_{k=0}^{+\infty} \frac{(-1)^k\left(\frac{t}{2}\right)^{2k}}{k!\Gamma(k+\alpha+1)} .$$

The definition of K_α extends by continuity to the case $\alpha \in \mathbb{N}$. K_α satisfies on $\mathbb{R}_+^*$ the ordinary differential equation

$$(1.18)\qquad t^2K_\alpha'' + tK_\alpha' - (t^2+\alpha^2)K_\alpha = 0 .$$

Note that, for $\alpha > 0$

$$(1.19)\qquad K_\alpha(t) = C_\alpha t^{-\alpha} + O(t^{\min(-\alpha+1,\alpha)}) \quad \text{as } t \to 0^+$$

and we have

$$(1.20)\qquad C_k = 2^{k-1}(k-1)! \;\; k \in \mathbb{N}^* \; ; C_{k+\frac{1}{2}} = \frac{\sqrt{\pi}(2k)!}{2^{k+\frac{1}{2}}k!} \quad k \in \mathbb{N} .$$

We set

$$(1.21)\qquad \varphi_{\lambda,y,\mu}(x) = \frac{1}{\lambda^{\frac{n-2}{2}}}\left(\frac{1}{|x-y|^{n-2}} - \frac{\mu^{\frac{n-2}{4}}K_{\frac{n-2}{2}}(\sqrt{\mu}|x-y|)}{C_{\frac{n-2}{2}}(\cancel{\sqrt{\mu}}|x-y|)^{\frac{n-2}{2}}}\right)$$

and

$$(1.22)\qquad V_{\lambda,y,\mu}(x) = U_{\lambda,y}(x) - \varphi_{\lambda,y,\mu}(x) .$$

A direct computation shows that

$$(1.23)\qquad -\Delta\varphi_{\lambda,y,\mu} + \mu\varphi_{\lambda,y,\mu} = \frac{\mu}{\lambda^{\frac{n-2}{2}}|x-y|^{n-2}}$$

whence

$$(1.24)\qquad -\Delta\bar{\alpha}V_{\lambda,y,\mu} + \mu\bar{\alpha}V_{\lambda,y,\mu} = (\bar{\alpha}U_{\lambda,y})^{\frac{n+2}{n-2}} + \mu\bar{\alpha}\left(U_{\lambda,y} - \frac{1}{\lambda^{\frac{n-2}{2}}|x-y|^{n-2}}\right) .$$

$\bar{\alpha}V_{\lambda,y,\mu}$ will turn out to be a suitable approximation of the concentrated solutions that we look for. We notice that in the special case $n = 3$ we are interested in throughout the sequel, $\frac{n+2}{n-2} = 5$, $\bar{\alpha} = 3^{1/4}$, $\frac{n-2}{2} = \frac{1}{2}$ and

$$(1.25)\qquad \frac{K_{1/2}(t)}{C_{1/2}} = \frac{e^{-t}}{\sqrt{t}} \quad \forall t > 0 .$$

The next section is devoted to the parametrization of the variational problem in a neighbourhood of the approximate solutions of (P) that we defined. The study of the parametrized problem will lead to the proof of the theorems in Section 3. The arguments require some precise integral estimates which are established in appendix.

2 Parametrization of the variational problem

We proceed to a suitable parametrization of the variational problem in a neighbourhood of the approximate solutions. For $\varepsilon > 0$, we set

$$(2.1) \qquad B_{\varepsilon,\mu} = \left\{ (\alpha, \lambda, y) \in \mathbb{R} \times \mathbb{R}_+^* \times \partial\Omega / \frac{\bar{\alpha}}{2} < \alpha < 2\bar{\alpha} \ , \ \lambda > \frac{\sqrt{\mu}}{\varepsilon} \right\}$$

and

$$(2.2) \qquad \mathcal{V}_{\varepsilon,\mu} = \left\{ u \in H^1(\Omega) / \exists (\lambda, y) \in \mathbb{R}_+^* \times \partial\Omega, \lambda > \frac{\sqrt{\mu}}{\varepsilon}, \ |\nabla(u - \bar{\alpha} V_{\lambda,y,\mu})|_2 < \varepsilon \right\} .$$

Note that according to the previous definitions, in the 3-dimensional case that we are interested in, we have

$$(2.3) \qquad V_{\lambda,y,\mu}(x) = \frac{\lambda^{1/2}}{(1+\lambda^2|x-y|^2)^{1/2}} - \frac{1-e^{-\sqrt{\mu}|x-y|}}{\lambda^{1/2}|x-y|} .$$

From now on, we assume that $\mu \geq \mu_0$, with μ_0 a strictly positive constant. We prove the following proposition :

Proposition 2.1 *There exists $\varepsilon_0 > 0$ such that for any ε, $0 < \varepsilon \leq \varepsilon_0$, and any $u \in \mathcal{V}_{\varepsilon,\mu}$, the infimum.*

$$\inf_{(\alpha,\lambda,y)\in B_{4\varepsilon,\mu}} |\nabla(u - \alpha V_{\lambda,y,\mu})|_2$$

is achieved at only one point, which lies in $B_{2\varepsilon,\mu}$.

A similar proposition is stated in [23], whose proof is modelled on [8, Proposition 7]. We indicate in Appendix A in which way the arguments of [23] have to be modified to conclude in the present case. (As we shall see further, the largeness of $\lambda > \frac{\sqrt{\mu}}{\varepsilon}$ and the smallness of $\frac{\sqrt{\mu}}{\lambda} < \varepsilon$ ensure that $V_{\lambda,y,\mu}$ given by (2.1) is a perturbation of $U_{\lambda,y}$ in $H^1(\Omega)$.)

For $\lambda \in \mathbb{R}_+^*$ and $y \in \partial\Omega$, we define

(2.4)

$$E_{\lambda,y,\mu} = \left\{ v \in H^1(\Omega) / \int_\Omega \nabla v . \nabla V_{\lambda,y,\mu} = \int_\Omega \nabla v . \nabla \frac{\partial V_{\lambda,y,\mu}}{\partial \lambda} = \int_\Omega \nabla v . \nabla \frac{\partial V_{\lambda,y,\mu}}{\partial \tau_i} = 0, i = 1, 2 \right\}$$

$(\tau_i)_{i=1,2}$ denoting an orthonormal system of coordinates on the tangent space to $\partial\Omega$ at y. For $\mu \geq \mu_0$, Proposition 2.1 induces a map from the open subset $\mathcal{V}_{\varepsilon_0,\mu}$ of $H^1(\Omega)$ to the manifold

$$\mathcal{M}_\mu = \left\{ (\alpha, \lambda, y, v) \in \mathbb{R} \times \mathbb{R}_+^* \times \partial\Omega \times H^1(\Omega) / (\alpha, \lambda, y) \in B_{2\varepsilon_0,\mu} \ , v \in E_{\lambda,y,\mu}, |\nabla v|_2 < \varepsilon \right\}$$

$(\alpha, \lambda, y) \in B_{2\varepsilon_0,\mu}$ being the unique point such that $\inf_{(\alpha,\lambda,y)\in B_{4\varepsilon_0,\mu}} |\nabla(u - \alpha V_{\lambda,y,\mu})|_2$ is achieved, and $v = u - \alpha V_{\lambda,y,\mu}$. It is easily seen that this map is open, and induces a diffeomorphism between $\mathcal{V}_{\varepsilon_0,\mu}$ and its image. This image contains, for $\eta_0 > 0$ small enough, the open subset

$$\mathcal{N}_\mu = \left\{(\alpha, \lambda, y, v) \in \mathbb{R}\times\mathbb{R}_+^* \times\partial\Omega\times H^1(\Omega)/|\alpha-\bar{\alpha}| < \eta_0\ ,\ \lambda > \frac{\sqrt{\mu}}{\eta_0}, v \in E_{\lambda,y,\mu}, |\nabla v|_2 < \eta_0\right\}.$$

Defining the functional

$$\begin{array}{ll} J_\mu : \mathcal{N}_\mu & \to \mathbb{R} \\ (\alpha, \lambda, y, v) & \mapsto I_\mu(\alpha V_{\lambda,y,\mu} + v) \end{array}$$

$(\alpha, \lambda, y, v) \in \mathcal{N}_\mu$ is a critical point of J_μ if and only if $u = \alpha V_{\lambda,y,\mu} + v \in H^1(\Omega)$ is a critical point of I_μ. The 0-homogenity of J_μ even leads to define the functional

$$\begin{array}{ll} K_\mu : \mathcal{O}_\mu & \to \mathbb{R} \\ (\lambda, y, v) & \mapsto I_\mu(V_{\lambda,y,\mu} + v) \end{array} \tag{2.5}$$

with

$$\mathcal{O}_\mu = \left\{(\lambda, y, v) \in \mathbb{R}_+^* \times \partial\Omega \times H^1(\Omega)/\lambda > \frac{\sqrt{\mu}}{\eta_0'}, v \in E_{\lambda,y,\mu}, |\nabla v|_2 < \eta_0'\right\} \tag{2.6}$$

and $\eta_0' > 0$ is a constant sufficiently small. $(\lambda, y, v) \in \mathcal{O}_\mu$ is a critical point of K_μ if and only if $u = V_{\lambda,y,\mu} + v \in H^1(\Omega)$ is a critical point of I_μ. Then, $\tilde{u} = (I_\mu(u))^{1/4}u$ satisfies

$$-\Delta\tilde{u} + \mu\tilde{u} = u^5 \text{ in } \Omega\ ;\ \frac{\partial\tilde{u}}{\partial\nu} = 0 \text{ on } \partial\Omega\ .$$

3 Proof of the theorems

This proof will follow from the study of the functional K_μ on $\mathcal{O}_\mu$, showing the existence, for μ large enough, of $(\lambda_\mu, y_\mu, v_\mu) \in \mathcal{O}_\mu$ a critical point of K_μ such that $u_\mu = V_{\lambda_\mu,y_\mu,\mu} + v_\mu > 0$ in Ω - whence a solution of (P).

For u, u' in $H^1(\Omega)$, we define the scalar product and the norm

$$\langle u, u'\rangle = \int_\Omega \nabla u.\nabla u' + \mu\int_\Omega uu' \qquad ||u|| = \langle u, u\rangle^{1/2}. \tag{3.1}$$

3.1 Minimization of K_μ with respect to v

In this subsection, we prove that for μ large enough, $\frac{\sqrt{\mu}}{\lambda}$ small enough, and $y \in \partial\Omega$, there exists $\bar{v} \in E_{\lambda,y,\mu}$ such that the partial derivative of K_μ with respect to v vanishes. Moreover, we estimate the norm of $\bar{v}$ in such a way that this term will appear as negligible in the expansion of K_μ. According to (1.3) and (2.5), we have

$$K_\mu(\lambda, y, v) = \frac{||V||^2 + 2\mu\int_\Omega Vv + ||v||^2}{(\int_\Omega(V + v)^6)^{1/3}}$$

with $V = V_{\lambda,y,\mu}$ for sake of simplicity. We write

$$\begin{aligned}\int_\Omega (V+v)^6 &= \int_\Omega V^6 + 6\int_\Omega V^5 v + 15\int_\Omega V^4 v^2 + O(|v|_{H^1}^3)\\ &= \int_\Omega V^6 + O(|v|_{H^1})\end{aligned}$$

as $|v|_{H^1}$ goes to zero, since V is bounded in L^6 as μ goes to infinity and $\frac{\sqrt{\mu}}{\lambda}$ goes to zero - see estimate (C.3) in appendix. We have also

$$\mu\left|\int_\Omega Vv\right| \leq \mu\left(\int_\Omega V^2\right)^{1/2}\left(\int_\Omega v^2\right)^{1/2} = O\left(\frac{\mu^{1/4}}{\lambda^{1/2}}\|v\|\right) \tag{3.2}$$

according to estimate (C.2). Then, in a neighbourhood of $v=0$, K_μ writes as

$$K_\mu(\lambda,y,v) = K_\mu(\lambda,y,0) + f_{\lambda,y,\mu}(v) + Q_{\lambda,y,\mu}(v) + R_{\lambda,y,\mu}(v) \tag{3.3}$$

with (omitting the indices λ, y, μ)

$$f(v) = 2\left(\int_\Omega V^6\right)^{-1/3}\left(\mu\int_\Omega Vv - \ell(V)\int_\Omega V^5 v\right) \tag{3.4}$$

$$\begin{aligned}Q(v) = \left(\int_\Omega V^6\right)^{-1/3}\Big[\|v\|^2 - 5\ell(V)\int_\Omega V^4v^2 &-4\mu\left(\int_\Omega V^6\right)^{-1}\left(\int_\Omega Vv\right)\left(\int_\Omega V^5 v\right)\\ &+8\left(\int_\Omega V^6\right)^{-1}\ell(V)\left(\int_\Omega V^5v\right)^2\Big]\end{aligned} \tag{3.5}$$

$$R(v) = O(\|v\|^2|v|_{H^1}) \tag{3.6}$$

where we set

$$\ell(V) = \left(\int_\Omega V^6\right)^{-1}\|V\|^2. \tag{3.7}$$

Moreover, we easily verify that

$$R'(v) = O(\|v\|^2) \qquad R''(v) = O(\|v\|) \tag{3.8}$$

in $H^1(\Omega)$ equipped with $\|\ \|$, uniformly with respect to $y \in \partial\Omega$, μ and λ, provided that μ is large enough and $\frac{\sqrt{\mu}}{\lambda}$ is small enough. Concerning the behaviour of the linear form f and the quadratic form Q, we have :

Lemma 3.1 *The norm of $f_{\lambda,y,\mu}$ as a linear form on $H^1(\Omega)$ equipped with the scalar product (3.1) verifies*

$$\|f_{\lambda,y,\mu}\| = O\left(\frac{1}{\lambda^{1/2}\mu^{1/4}} + \frac{\mu^{1/2}}{\lambda}\right) \tag{3.9}$$

uniformly with respect to $y \in \partial\Omega$, μ, λ, as μ is large enough and $\frac{\sqrt{\mu}}{\lambda}$ small enough.

Lemma 3.2 *The quadratic form $Q_{\lambda,y,\mu}$ is coercive on $H^1(\Omega)$ equipped with the scalar product (3.1), as μ is large enough and $\frac{\sqrt{\mu}}{\lambda}$ is small enough, with a modulus of coercivity bounded from below by a striclty positive constant independent of $y \in \partial\Omega, \lambda, \mu$.*

Lemma 3.2 is proved in Appendix B. Before proving Lemma 3.1, we note that the derivative of K_μ with respect to v vanishes if and only if $g + Av + r(v) = 0$ in $E_{\lambda,y,\mu}$, with $g, r(v) \in E_{\lambda,y,\mu}$, $A \in \mathcal{L}(E_{\lambda,y,\mu})$ such that

$$f(v) = \langle g, v\rangle \quad Q(v) = \frac{1}{2}\langle Av, v\rangle \quad R'(v) = \langle r(v), .\rangle \; .$$

Applying the implicit theorem to the map

$$\begin{aligned} F : E_{\lambda,y,\mu} \times E_{\lambda,y,\mu} &\to E_{\lambda,y,\mu} \\ (h, v) &\mapsto h + Av + r(v) \end{aligned}$$

which verifies $F(0,0) = 0$, $\frac{\partial F}{\partial v}(h, v) = A + r'(v)$ invertible in a neighourhood of $(0,0)$ because of Lemma 3.2 and (3.8), we obtain :

Proposition 3.1 *There exists $\delta_0 > 0, \delta_0' > 0$ such that, for μ large enough, there exists a smooth map*

$$\begin{aligned} \{(\lambda, y) \in \mathbb{R}_+^* \times \partial\Omega / \tfrac{\sqrt{\mu}}{\lambda} < \delta_0\} &\to E_{\lambda,y,\mu} \\ (\lambda, y) &\mapsto \bar{v}_\mu(\lambda, y) \end{aligned}$$

such that $\bar{v}_\mu(\lambda, y)$ is the unique point $v \in E_{\lambda,y,\mu}$, $\|v\| < \delta_0'$, satisfying $\frac{\partial K_\mu}{\partial v}(\lambda, y, \bar{v}_\mu(\lambda, y)) = 0$ in $T_{\lambda,y,\bar{v}_\mu}\mathcal{O}_\mu$. Moreover

(3.10) $$\|\bar{v}_\mu(\lambda, y)\| = O(\frac{1}{\lambda^{1/2}\mu^{1/4}})$$

uniformly with respect to $y \in \partial\Omega$ as μ, λ go to infinity, $\frac{\sqrt{\mu}}{\lambda} < \delta_0$.

We turn now to the proof of Lemma 3.1. Estimates (C.1) (C.2) (C.3) of Appendix C show that ($V = V_{\lambda,y,\mu}$)

(3.11) $$\int_\Omega V^6 = \frac{\pi^2}{8} + O(\frac{\sqrt{\mu}}{\lambda}) \qquad \ell(V) = 3 + O(\frac{\mu^{1/4}}{\sqrt{\lambda}})$$

as μ goes to infinity and $\frac{\sqrt{\mu}}{\lambda}$ goes to zero. We write ($\varphi = \varphi_{\lambda,y,\mu}, U = U_{\lambda,y}$)

(3.12) $$\int_\Omega V^5 v = \int_\Omega (U - \varphi)^5 v = \int_\Omega U^5 v + O(\int_\Omega U^4 \varphi|v| + \int_\Omega \varphi^5|v|) \; .$$

On one hand, Hölder's inequality, Sobolev embedding, (C.49) and (C.12) imply that

(3.13) $$\int_\Omega U^4\varphi|v| \le C\left(\int_\Omega U^{24/5}\varphi^{6/5}\right)^{5/6} |v|_{H^1} = O\left(\frac{\sqrt{\mu}}{\lambda}|v|_{H^1}\right)$$

and

$$\int_\Omega \varphi^5|v| \le C\left(\int_\Omega \varphi^6\right)^{5/6}|v|_{H^1} = O\left(\frac{\mu^{5/4}}{\lambda^{5/2}}|v|_{H^1}\right)\,. \tag{3.14}$$

On the other hand, from (1.24) we deduce that

$$\int_\Omega U^5 v = \int_\Omega \frac{1}{3}\left(-\Delta V + \mu V - \mu(U - \frac{1}{\lambda^{1/2}|.-y|})\right) v$$

whence, integrating by parts and using the fact that $v \in E_{\lambda,y,\mu}$

$$\int_\Omega U^5 v = \frac{1}{3}\left(-\int_{\partial\Omega}\frac{\partial V}{\partial \nu}v + \mu\int_\Omega Vv - \mu\int_\Omega (U - \frac{1}{\lambda^{1/2}|\cdot - y|})v\right)\,. \tag{3.15}$$

As a consequence, (3.4) (3.11) (3.12) (3.13) (3.14), imply that

$$f(v) = O\left(\left|\int_{\partial\Omega}\frac{\partial V}{\partial\nu}v\right| + \frac{\mu^{5/4}}{\lambda}\left|\int_\Omega Vv\right| + \mu\left|\int_\Omega (U - \frac{1}{\lambda^{1/2}|\cdot - y|})v\right| + \frac{\sqrt{\mu}}{\lambda}|v|_{H^1}\right)\,. \tag{3.16}$$

We have

$$\left|\int_\Omega \frac{\partial V}{\partial\nu}v\right| \le C\left(\int_{\partial\Omega}|\frac{\partial V}{\partial\nu}|^{4/3}\right)^{3/4}|v|_{H^1} = O\left(\frac{|v|_{H^1}}{\lambda^{1/2}\mu^{1/4}}\right) \tag{3.17}$$

using Hölder's inequality, the embedding of $H^1(\Omega)$ in $L^4(\partial\Omega)$ and (C.52), and

$$\mu\left|\int_\Omega\left(U - \frac{1}{\lambda^{1/2}|\cdot - y|}\right)v\right| \le \mu\left(\int_\Omega\left|U - \frac{1}{\lambda^{1/2}|\cdot - y|}\right|^{6/5}\right)^{5/6}|v|_{H^1} = O\left(\frac{\mu}{\lambda^2}|v|_{H^1}\right) \tag{3.18}$$

using Schwarz inequality, Sobolev embedding and (C.51). (3.16) (3.17) (3.18) (3.2) show that (3.9) holds, hence Lemma 3.1.

3.2 Proof of Theorem 1.1 and Proposition 1.1

Let us state the following proposition :

Proposition 3.2 *Let u_μ be a low energy solution of (P), i.e. u_μ solves (P) and $I_\mu(u_\mu) < \frac{3\pi^{4/3}}{4}$. For μ large enough, there exist $(\alpha_\mu, \lambda_\mu, y_\mu) \in \mathbb{R}\times\mathbb{R}_+^*\times\partial\Omega, \alpha_\mu \to 3^{1/4}, \lambda_\mu \sim \frac{(\max_{\overline{\Omega}} u_\mu)^2}{\sqrt{3}}, \frac{\sqrt{\mu}}{\lambda_\mu} \to 0$ as $\mu\to\infty$, such that*

$$u_\mu = \alpha_\mu(V_{\lambda_\mu,y_\mu,\mu} + \bar{v}_\mu(\lambda_\mu, y_\mu)). \tag{3.19}$$

The proof of this proposition is delayed until the end of this subsection.

Proof of Theorem 1.1. According to Proposition 3.1, there exist $\bar{\mu} > 0$ and $\bar{\eta} > 0$ such that the function

$$\widetilde{K}_\mu(\lambda, y) = K_\mu(\lambda, y, \bar{v}_\mu(\lambda, y)) \tag{3.20}$$

is well defined, for $\mu > \bar{\mu}$, on $\{(\lambda, y) \in \mathbb{R}_+^* \times \partial\Omega / \frac{\sqrt{\mu}}{\lambda} < \bar{\eta}\}$. From (3.3), (3.5) (3.6), (3.9) and (3.10), we deduce that

$$\widetilde{K}_\mu(\lambda, y) = K_\mu(\lambda, y, 0) + O\left(\frac{1}{\lambda\sqrt{\mu}} + \frac{\mu}{\lambda^2}\right) \tag{3.21}$$

and estimate (C.4) provides us with the expansion

(3.22)

$$\widetilde{K}_\mu(\lambda, y) = 2\pi^{1/3}\left(\frac{3\pi}{8} + 2\frac{\sqrt{\mu}}{\lambda} - \frac{H(y)}{\lambda}(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma)\right) + O\left(\frac{1}{\lambda\sqrt{\mu}} + \frac{\mu}{\lambda^2} + \frac{\sqrt{\mu}}{\lambda^2}\mathrm{Log}\frac{\lambda}{\sqrt{\mu}}\right) .$$

Choosing $\bar{y} \in \partial\Omega$ such that $H(\bar{y}) = \max_{y \in \partial\Omega} H(y) = \bar{H}$, and $\bar{\lambda}$ such that $\mathrm{Log}\bar{\lambda} = \frac{2\sqrt{\mu}}{H(\bar{y})} + \mathrm{Log}2\sqrt{\mu} + \gamma + \frac{1}{2}$, we obtain

$$\widetilde{K}_\mu(\bar{\lambda}, \bar{y}) = \frac{3\pi^{4/3}}{4} - \frac{\pi^{1/3}}{\sqrt{\mu}}\left(\bar{H} + O(\frac{1}{\sqrt{\mu}})\right)\exp\left(-\frac{2\sqrt{\mu}}{\bar{H}} - \gamma - \frac{1}{2}\right) . \tag{3.23}$$

On the other hand, if u_μ is a minimizer of I_μ, for μ large enough Proposition 3.2. applies and, according to (3.19) (3.20) (3.21) and (3.22)

$$I_\mu(u_\mu) = 2\pi^{1/3}\left(\frac{3\pi}{8} + 2\frac{\sqrt{\mu}}{\lambda_\mu} - \frac{H(y_\mu)}{\lambda_\mu}(\mathrm{Log}\frac{\lambda_\mu}{2\sqrt{\mu}} - \gamma + \frac{1}{2})\right) + O\left(\frac{1}{\lambda_\mu\sqrt{\mu}} + \frac{\mu}{\lambda_\mu^2} + \frac{\sqrt{\mu}}{\lambda_\mu^2}\mathrm{Log}\frac{\lambda_\mu}{\sqrt{\mu}}\right) .$$

As a consequence, $H(y_\mu) \geq 0$ and, minimizing this expansion with respect to λ_μ, $\sqrt{\mu}/\lambda_\mu$ going to zero, we find

$$I_\mu(u_\mu) \geq \frac{3\pi^{4/3}}{4} - \frac{\pi^{1/3}}{\sqrt{\mu}}\left(H(y_\mu) + O(\frac{1}{\sqrt{\mu}})\right)\exp\left(-\frac{2\sqrt{\mu}}{H(y_\mu)} - \gamma - \frac{1}{2}\right) . \tag{3.24}$$

Necessarily, $I_\mu(u_\mu) \leq \widetilde{K}_\mu(\bar{\lambda}, \bar{y})$, whence, in view of (3.23) and (3.24)

$$\frac{H(y_\mu)}{\bar{H}} + O(\frac{1}{\sqrt{\mu}}) \geq \exp\left(-2\sqrt{\mu}(\frac{1}{\bar{H}} - \frac{1}{H(y_\mu)})\right) .$$

This proves that

$$H(y_\mu) = \bar{H} + O(\frac{1}{\mu}) \quad \text{as } \mu \to +\infty . \tag{3.25}$$

Moreover, as λ_μ minimizes $\widetilde{K}_\mu(\lambda, y_\mu)$ with respect to λ, we obtain using (3.22)

$$\text{Log}\lambda_\mu = \frac{2\sqrt{\mu}}{\bar{H}} + \text{Log}2\sqrt{\mu} + \gamma + \frac{1}{2} + O(\frac{1}{\sqrt{\mu}}) \ . \tag{3.26}$$

(1.6) follows from (3.25) (3.26) and (3.22). (1.7) follows from (3.26) and Proposition 3.2.

Proof of Proposition 1.1. We just proved that for μ large enough,

$$I_\mu(u) = \frac{\int_\Omega |\nabla u|^2 + \mu \int_\Omega u^2}{(\int_\Omega u^6)^{1/3}} \geq I_\mu(u_\mu) = \frac{3\pi^{4/3}}{4} - \eta \qquad \forall u \in H^1(\Omega)$$

with

$$\eta = \left(\frac{\pi^{1/3}\bar{H}}{\sqrt{\mu}} + O(\frac{1}{\mu})\right) \exp\left(-\frac{2\sqrt{\mu}}{\bar{H}} - \frac{1}{2} - \gamma\right) \ .$$

Then

$$\text{Log}\eta = -\frac{2\sqrt{\mu}}{\bar{H}} - \text{Log}\frac{\sqrt{\mu}}{\bar{H}} - \frac{1}{2} - \gamma + \text{Log}\pi^{1/3} + O(\frac{1}{\sqrt{\mu}}) \ .$$

This implies that $\sqrt{\mu} = \frac{\bar{H}}{2}|\text{Log}\eta|(1+\beta(\eta))$, with $\beta(\eta) = o(1)$ as η goes to zero, and we find

$$\beta(\eta) = \frac{1}{\text{Log}\eta}\left(\text{Log}|\text{Log}\eta| + \frac{1}{2} + \gamma - \text{Log}2\pi^{1/3} + O(\frac{1}{|\text{Log}\eta|})\right)$$

hence (1.9).

Proof of Proposition 3.2. Let u_μ be a low energy solution of (P). i.e. u_μ solves (P) and $I_\mu(u_\mu) < \frac{3\pi^{4/3}}{4}$. Since

$$\int_\Omega |\nabla u_\mu|^2 + \mu \int_\Omega u_\mu^2 = \int_\Omega u_\mu^6 \tag{3.27}$$

there exists $x \in \Omega$ such that $u_\mu(x) \geq \mu^{1/4}$, and $\sup_{\bar{\Omega}} u_\mu$ goes to infinity as μ goes to infinity. From [4, Lemma 2.2], we know that this supremum is achieved at only one point $y'_\mu \in \partial\Omega$, and

$$|\nabla(u_\mu - 3^{1/4}U_{\lambda'_\mu, y'_\mu})|_2 \to 0 \quad \text{as } \mu \to \infty \tag{3.28}$$

with

$$\lambda'_\mu = \left(\frac{u_\mu(y'_\mu)}{3^{1/4}}\right)^2 \qquad\qquad \frac{\sqrt{\mu}}{\lambda'_\mu} \to 0 \quad \text{as } \mu \to 0 \ . \tag{3.29}$$

(3.28) implies that $|\nabla(u_\mu - 3^{1/4}V_{\lambda'_\mu, y'_\mu, \mu})|_2 \to 0$ as $\mu \to \infty$, since

$$|\nabla(U_{\lambda,y} - V_{\lambda,y,\mu})|_2 = |\nabla\varphi_{\lambda,y,\mu}|_2 \to 0 \text{ as } \mu \to \infty \ , \frac{\sqrt{\mu}}{\lambda} \to 0 \tag{3.30}$$

as estimate (C.43) shows. Therefore, Proposition 2.1. applies and we may write

$$u_\mu = \alpha_\mu (V_{\lambda_\mu, y_\mu, \mu} + v_\mu) \tag{3.31}$$

with $\frac{3^{1/4}}{2} < \alpha_\mu < 2.3^{1/4}, y_\mu \in \partial\Omega, v_\mu \in E_{\lambda_\mu, y_\mu, \mu}, |\nabla v_\mu|_2 \to 0$ and $\frac{\sqrt{\mu}}{\lambda_\mu} \to 0$ as $\mu \to \infty$. Actually, in view of (3.28) (3.30) and (3.31), we have

$$|\nabla \left(\alpha_\mu U_{\lambda_\mu, y_\mu} - 3^{1/4} U_{\lambda'_\mu, y'_\mu}\right)|_2 \to 0 \left(\text{resp. } |\nabla \left(\alpha_\mu V_{\lambda_\mu, y_\mu, \mu} - 3^{1/4} V_{\lambda'_\mu, y_\mu, \mu}\right)|_2 \to 0\right)$$

as $\mu \to \infty$ whence, according to [23, Lemma A.1] (resp. Lemma A.1 in appendix)

$$\alpha_\mu \to 3^{1/4} \quad \lambda_\mu/\lambda'_\mu \to 1 \quad \lambda_\mu \lambda'_\mu |y_\mu - y'_\mu|^2 \to 0 \quad \text{as } \mu \to \infty\,. \tag{3.32}$$

Consequently

$$\lambda_\mu \sim \frac{(\max_{\bar\Omega} u)^2}{\sqrt{3}} \qquad \frac{\sqrt{\mu}}{\lambda_\mu} \to 0 \quad \text{as } \mu \to \infty\,.$$

Note that we have also $\|v_\mu\| \to 0$ as $\mu \to \infty$. Indeed, $I_\mu(u_\mu) < \frac{3\pi^{4/3}}{4}$ and (3.27) imply that

$$\int_\Omega |\nabla u_\mu|^2 + \mu \int_\Omega u_\mu^2 < \frac{3^{3/2}\pi^2}{8}\,.$$

Then, (3.31) and estimate (C.1) show that

$$\alpha_\mu^2 \left(\frac{3\pi^2}{8} + o(1) + \int_\Omega |\nabla v_\mu|^2\right) + \mu \int_\Omega u_\mu^2 < \frac{3^{3/2}\pi^2}{8}\,.$$

We know that $\alpha_\mu \to 3^{1/4}$ as $\mu \to \infty$, so

$$\mu \int_\Omega u_\mu^2 = o(1)\,.$$

Estimate (C.2) shows that $\mu \int_\Omega V^2_{\lambda_\mu, y_\mu, \mu} = o(1)$, therefore $\mu \int_\Omega v_\mu^2 = o(1)$ and $\|v_\mu\| \to 0$ as $\mu \to \infty$. Then, Proposition 3.1 shows that $v_\mu = \bar v_\mu(\lambda_\mu, y_\mu)$. This completes the proof of Proposition 3.2.

3.3 Proof of Theorem 1.2

Assuming that μ is large enough and $\frac{\sqrt{\mu}}{\lambda}$ small enough, we look for critical points of $\widetilde{K}_\mu(\lambda, y) = K_\mu(\lambda, y, \bar v(\lambda, y))$, which provide us with critical points of I_μ. Such critical points are actually solutions of (P) :

Proposition 3.3 *There exist $\mu^* > 0$ and $\eta^* > 0$ such that for any $\mu > \mu^*, \widetilde{K}_\mu$ is well defined on $\mathcal{K}_\mu = \{(\lambda, y) \in \mathbb{R}_+^* \times \partial\Omega / \frac{\sqrt{\mu}}{\lambda} < \eta^*\}$, and if $(\lambda, y) \in \mathcal{K}_\mu$ is a critical point of $\widetilde{K}_\mu$, there exist $\alpha > 0$ such that $u = \alpha(V_{\lambda, y, \mu} + \bar v_\mu(\lambda, y))$ is a solution of (P).*

Proof : If (λ, y) is a critical point of $\widetilde{K}_\mu$, $\tilde{u} = V_{\lambda,y,\mu} + \bar{v}_\mu(\lambda, y)$ is a critical point of I_μ, i.e. there exists $\omega \in \mathbb{R}$ such that

$$-\Delta\tilde{u} + \mu\tilde{u} = \omega\tilde{u}^5 \text{ in } \Omega \; ; \frac{\partial\tilde{u}}{\partial\nu} = 0 \text{ on } \partial\Omega \; . \tag{3.33}$$

Note that $\tilde{u} \not\equiv 0$, provided that μ is large enough and $\frac{\sqrt{\mu}}{\lambda}$ is small enough, since $\|\bar{v}_\mu(\lambda, y)\| \to 0$ and $\|V_{\lambda,y,\mu}\|^2 \to \frac{3\pi^2}{8}$ (see estimates (C.1) (C.2)) as $\mu \to \infty$ and $\frac{\sqrt{\mu}}{\lambda} \to 0$. Multiplying the equation by $\tilde{u}$ and integrating on Ω, we find $\omega = 3 + o(1)$ as μ goes to infinity and $\frac{\sqrt{\mu}}{\lambda}$ goes to zero, because of (C.1) (C.2) (C.3) and (3.10). Setting $u = \omega^{1/4}\tilde{u}$, u satisfies

$$-\Delta u + \mu u = u^5 \text{ in } \Omega \; ; \frac{\partial u}{\partial\nu} = 0 \quad \text{on } \Omega \; . \tag{3.34}$$

Finally, let us show that $u > 0$ in Ω. We multiply (3.33) by $u^- = \max(0, -u)$ and integrate on Ω. This yields

$$\|u^-\|^2 = |u^-|_6^6$$

On the other hand, Sobolev embedding theorem ensures that

$$\|u^-\|^2 \geq C|u^-|_6^2$$

with C a strictly positive constant. Thus, either $u^- \equiv 0$, or $|u^-|_6 \geq C^{1/4}$. Note that $0 \leq u^- \leq \omega^{1/4}(\varphi_{\lambda,y,\mu} + |\bar{v}_\mu(\lambda, y)|)$. As $\varphi_{\lambda,y,\mu}$ and $\bar{v}_\mu(\lambda, y)$ go to zero in $L^6(\Omega)$ as μ goes to infinity and $\frac{\sqrt{\mu}}{\lambda}$ goes to zero, $u^- \equiv 0$ provided that $\mu > \mu^*$, $\frac{\sqrt{\mu}}{\lambda} < \eta^*$, with μ^* and η^* two suitably chosen strictly positive constants. The strong maximum principle then implies that $u > 0$ in Ω, and u is a solution of (P).

Proof of Theorem 1.2. It is equivalent to consider, instead of $\widetilde{K}_\mu$

$$L_\mu = \frac{1}{2\pi^{1/3}}\widetilde{K}_\mu - \frac{3\pi}{8} \; . \tag{3.35}$$

Let $y_0 \in \partial\Omega$ be a strict local maximum point of H, $H(y_0) > 0$. Then

$$\exists r > 0 \; s.t. \; \forall y \in V_r = \{y \in \partial\Omega, |y - y_0| < r\} \; , \; y \neq y_0 \Rightarrow H(y) < H(y_0) \; . \tag{3.36}$$

Let

$$\mathcal{L}_\mu = \left\{(\lambda, y)/y \in V_r, \frac{\sqrt{\mu}}{H(y_0)} < \text{Log}\lambda < \frac{3\sqrt{\mu}}{H(y_0)}\right\} \; .$$

Note that for μ large enough, $\mathcal{L}_\mu \subset \mathcal{K}_\mu$. On the compact set $\overline{\mathcal{L}_\mu}$, the continuous function L_μ achieves its infimum. We claim that this infimum is not achieved on $\partial\mathcal{L}_\mu$. Indeed, in view of (3.22) we have on $\mathcal{L}_\mu$ the following expansion

$$\left|L_\mu(\lambda, y) - \left(2\frac{\sqrt{\mu}}{\lambda} - \frac{H(y)}{\lambda}(\text{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma)\right)\right| \leq \frac{A}{\lambda\sqrt{\mu}} \tag{3.37}$$

for μ large enough, and A is some positive constant. Let λ_0 be such that

$$\mathrm{Log}\lambda_0 = \frac{2\sqrt{\mu}}{H(y_0)} + \mathrm{Log}2\sqrt{\mu} + \gamma + \frac{1}{2} + \frac{A}{H(y_0)\sqrt{\mu}} \, .$$

For μ large enough, $(\lambda_0, y_0) \in \mathcal{L}_\mu$ and

$$L_\mu(\lambda_0, y_0) \le -\frac{H(y_0)}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{H(y_0)} - \gamma - \frac{1}{2} - \frac{A}{H(y_0)\sqrt{\mu}}\right) \, . \tag{3.38}$$

On the other hand, let $(\lambda, y) \in \partial\mathcal{L}_\mu$. Three cases are possible :

(i) $\mathrm{Log}\lambda = \frac{\sqrt{\mu}}{H(y_0)}$. Then, (3.37) implies that for μ large enough

$$L_\mu(\lambda, y) \ge \sqrt{\mu}\exp(-\frac{\sqrt{\mu}}{H(y_0)}) > L_\mu(\lambda_0, y_0).$$

(ii) $\mathrm{Log}\lambda = \frac{3\sqrt{\mu}}{H(y_0)}$. In this case (3.37) yields

$$L_\mu(\lambda, y) \ge -\sqrt{\mu}\exp(-\frac{3\sqrt{\mu}}{H(y_0)}) > L_\mu(\lambda_0, y_0) \qquad \text{for } \mu \text{ large enough.}$$

(iii) $|y - y_0| = r$. There exists $\delta > 0$ such that $H(y) < H(y_0) - \delta$, so that, using (3.37)

$$L_\mu(\lambda, y) \ge \frac{1}{\lambda}\left(2\sqrt{\mu} - (H(y_0) - \delta)(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma) - \frac{A}{\sqrt{\mu}}\right) \, .$$

For $\mathrm{Log}\lambda \in \left(\frac{\sqrt{\mu}}{H(y_0)}, \frac{3\sqrt{\mu}}{H(y_0}\right)$, the right hand side achieves its minimimum at $\bar\lambda$ such that $\mathrm{Log}\bar\lambda = \frac{1}{H(y_0)-\delta}(2\sqrt{\mu} - \frac{A}{\sqrt{\mu}}) + \mathrm{Log}2\sqrt{\mu} + \gamma + \frac{1}{2}$. Therefore

$$L_\mu(\lambda, y) \ge -\frac{H(y_0) - \delta}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{H(y_0) - \delta} - \gamma - \frac{1}{2} + \frac{A}{(H(y_0) - \delta)\sqrt{\mu}}\right) > L_\mu(\lambda_0, y_0)$$

for μ large enough.

Consequently, a point (λ_μ, y_μ) at which L_μ achieves its minimum in $\overline{\mathcal{L}_\mu}$ lies in $\mathcal{L}_\mu$ for μ large enough, and is a critical point of $\widetilde{K}_\mu$. According to Proposition 3.3, there exist α_μ close to $3^{1/4}$ such that $u_\mu = \alpha_\mu(V_{\lambda_\mu, y_\mu, \mu} + \bar{v}_\mu(\lambda_\mu, y_\mu))$ is a solution of (P). Moreover, in view of (3.35)

$$I_\mu(u_\mu) = \widetilde{K}_\mu(\lambda_\mu, y_\mu) = \frac{3\pi^{4/3}}{4} + 2\pi^{1/3} L_\mu(\lambda_\mu, y_\mu)$$

and

$$L_\mu(\lambda_\mu, y_\mu) \le L_\mu(\lambda_0, y_0) < 0$$

as (3.38) shows. Then, u_μ is a low energy solution of (P). Noticing that $|y_\mu - y_0| < r$, and that r may be chosen as small as desired as μ goes to infinity, we can impose y_μ to go to y_0 as μ goes to infinity, and the proof of Theorem 1.2 is complete.

3.4 Proof of Theorem 1.3

Let $a > 0$ be an isolated critical value of H on $\partial\Omega$, such that $H_*(H_{a-\delta}, H_{a+\delta})$ is nontrivial for any $\delta > 0$ sufficiently small. We fix $c > 0$ such that $a > c$, and we set

$$\mathcal{L}'_\mu = \left\{ (\lambda, y) \in \mathbb{R}_+^* \times \partial\Omega / H(y) > c \,,\, \frac{\sqrt{\mu}}{\bar{H}} < \mathrm{Log}\lambda < \frac{3\sqrt{\mu}}{c} \right\}$$

with $\bar{H} = \max_{\partial\Omega} H$. We note that $\mathcal{L}'_\mu \subset \mathcal{K}_\mu$ for μ large enough. Therefore, a critical point of L_μ in $\mathcal{L}'_\mu$ provides us, through Proposition 3.3, with a solution of (P). The strategy consists in proving that the difference of topology between the level sets of H induces the existence of such a critical point. Noticing that the expansion (3.37) holds on $\mathcal{L}'_\mu$, we proceed as follows :

(i) We compute

$$\min_{\partial\mathcal{L}'_\mu} L_\mu \geq -\frac{c}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{c} - \gamma - \frac{1}{2} + \frac{A}{c\sqrt{\mu}} \right) = b_\mu \,. \tag{3.39}$$

(ii) For $y \in H_0 = H^+ = \{y \in \partial\Omega, H(y) > 0\}$, we set

$$\mathrm{Log}\lambda(y) = \frac{2\sqrt{\mu}}{H(y)} + \mathrm{Log}2\sqrt{\mu} + \gamma + \frac{1}{2} + \frac{A}{H(y)\sqrt{\mu}} \,. \tag{3.40}$$

$\lambda(y)$ realizes the minimum of $(\lambda, y) \mapsto 2\frac{\sqrt{\mu}}{\lambda} - \frac{H(y)}{\lambda}(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma) + \frac{A}{\lambda\sqrt{\mu}}$ with respect to $\lambda > 0$. We notice that for μ large enough, $(\lambda(y), y) \in \mathcal{L}'_\mu$ for any $y \in H_c$. Then, for $a - \delta > c$, we prove the following properties :

$$\forall y \in H_{a-\delta} \qquad L_\mu(\lambda(y), y) < a'_{\mu,\delta} \tag{3.41}$$

with

$$a'_{\mu,\delta} = -\frac{a-\delta}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{a-\delta} - \gamma - \frac{1}{2} - \frac{A}{(a-\delta)\sqrt{\mu}} \right) \;;$$

$$\forall (\lambda, y) \in \mathcal{L}'_\mu \qquad L_\mu(\lambda, y) < a'_{\mu,\delta} \text{ implies that } y \in H_{a-2\delta} \;; \tag{3.42}$$

$$\forall (\lambda, y) \in \mathcal{L}'_\mu \qquad L_\mu(\lambda, y) < a''_{\mu,\delta} \text{ implies that } y \in H_{a+\delta} \tag{3.43}$$

with

$$a''_{\mu,\delta} = -\frac{a+\delta}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{a+\delta} - \gamma + \frac{1}{2} + \frac{A}{(a+\delta)\sqrt{\mu}} \right) \,.$$

We are now able to conclude. We first notice that for μ large enough, $a''_{\mu,\delta} < a'_{\mu,\delta} < b_\mu$. Let us assume that L_μ has no critical value between $a''_{\mu,\delta}$ and $a'_{\mu,\delta}$. Then, using the decreasing flow associated to the gradient of L_μ, the level set $L_\mu^{a'_{\mu,\delta}} \cap \mathcal{L}'_\mu$ retracts by deformation onto the level set $L_\mu^{a''_{\mu,\delta}} \cap \mathcal{L}'_\mu$ (since $a'_{\mu,\delta} < b_\mu$, we know that the orbits starting from $L_\mu^{a'_{\mu,\delta}} \cap \mathcal{L}'_\mu$ do not intersect $\partial\mathcal{L}'_\mu$). In view of (3.41), we consider the subset of $L_\mu^{a'_{\mu,\delta}} \cap \mathcal{L}'_\mu$ defined as

$$\{(\lambda(y), y)/y \in H_{a-\delta}\}$$

and follow its deformation along the flow. According to (3.42), y remains in $H_{a-2\delta}$, and according to (3.43) $y \in H_{a+\delta}$ as $(\lambda, y) \in L_\mu^{a''_{\mu,\delta}} \cap \mathcal{L}'_\mu$. Since a is an isolated critical value of H, $H_{a-\delta}$ is a strong retract of $H_{a-2\delta}$ for δ small enough. Then, the composition of the flow with the projection $(\lambda, y) \mapsto y$ onto $\partial\Omega$ and the retraction of $H_{a-2\delta}$ into $H_{a-\delta}$ provides us with a continuous deformation of $H_{a-\delta}$ into $H_{a+\delta}$, a contradiction with $H_*(H_{a-\delta}, H_{a+\delta}) \neq 0$. Consequently, L_μ has a critical value between $a''_{\mu,\delta}$ and $a'_{\mu,\delta}$.

If all the corresponding critical points $(\lambda, y) \in \mathcal{L}'_\mu$ were such that $y \in H_{a+\delta}$, the previous argument would still apply, that is it would be possible to deform $H_{a-\delta}$ into $H_{a+\delta}$. Therefore, L_μ has a critical point (λ_μ, y_μ) such that $a''_{\mu,\delta} \leq L_\mu(\lambda_\mu, y_\mu) < a'_{\mu,\delta}$, and $H(y_\mu) \leq a + \delta$. On the other hand, (3.42) implies that $H(y_\mu) > a - 2\delta$. As μ goes to infinity, δ may be chosen as small as desired, so that, for μ large enough, L_μ has a critical point (λ_μ, y_μ) such that $H(y_\mu)$ goes to a as μ goes to infinity. Hence, according to Proposition 3.3, the existence of a solution u_μ of (P) which, up to a subsequence, concentrates at a point $y_0 \in \partial\Omega$ such that $H(y_0) = a$ as μ goes to infinity. (3.35) and $L_\mu(\lambda_\mu, y_\mu) \leq a'_{\mu,\delta} < 0$ show that u_μ is a low energy solution.

To complete the proof of Theorem 1.3, it only remains to prove assertions (3.39) (3.41) (3.42) (3.43).

Proof of (3.39). Let $(\lambda, y) \in \partial\mathcal{L}'_\mu$. Three cases are possible :

(i) $\text{Log}\lambda = \frac{\sqrt{\mu}}{\bar{H}}$. Then, using (3.37), we have

$$L_\mu(\lambda, y) \geq e^{-\frac{\sqrt{\mu}}{\bar{H}}} \left((2 - \frac{H(y)}{\bar{H}})\sqrt{\mu} + H(y)(\text{Log}2\sqrt{\mu} - \frac{1}{2} + \gamma) - \frac{A}{\sqrt{\mu}} \right) \geq 0$$

for μ large enough.

(ii) $\text{Log}\lambda = \frac{3\sqrt{\mu}}{c}$. In this case

$$L_\mu(\lambda, y) \geq e^{\frac{-3\sqrt{\mu}}{c}} \left((2 - \frac{3H(y)}{c})\sqrt{\mu} + H(y)(\text{Log}2\sqrt{\mu} - \frac{1}{2} + \gamma) - \frac{A}{\sqrt{\mu}} \right) \geq -\frac{3\bar{H}}{c}\sqrt{\mu}e^{-\frac{3\sqrt{\mu}}{c}}$$

for μ large enough.

(iii) $H(y) = c$. We have

$$L_\mu(\lambda, y) \geq \frac{1}{\lambda} \left(2\sqrt{\mu} - c(\text{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma) - \frac{A}{\sqrt{\mu}} \right) .$$

The right hand side achieves its minimum, for $\lambda \in (e^{\frac{\sqrt{\mu}}{H}}, e^{\frac{3\sqrt{\mu}}{c}})$, at a point $\bar{\lambda}$ such $\mathrm{Log}\bar{\lambda} = \frac{2\sqrt{\mu}}{c} + \mathrm{Log}2\sqrt{\mu} + \gamma + \frac{1}{2} - \frac{A}{c\sqrt{\mu}}$. Therefore, if $(\lambda, y) \in \partial\mathcal{L}'_\mu$ with $H(y) = c$

$$L_\mu(\lambda, y) \geq -\frac{c}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{c} - \gamma - \frac{1}{2} + \frac{A}{c\sqrt{\mu}}\right) = b_\mu$$

and (3.39) holds, provided that μ is large enough.

Proof of (3.41). According to (3.37) and (3.40), and assuming that $H(y) > a - \delta$, we have

$$L_\mu(\lambda(y), y) \leq -\frac{H(y)}{\lambda(y)} < -\frac{a-\delta}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{a-\delta} - \gamma - \frac{1}{2} - \frac{A}{(a-\delta)\sqrt{\mu}}\right) = a'_{\mu.\delta} \, .$$

Proof of (3.42). From (3.37), we know that

$$L_\mu(\lambda, y) \geq \frac{1}{\lambda}\left(2\sqrt{\mu} - H(y)(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma) - \frac{A}{\sqrt{\mu}}\right) \qquad \forall (\lambda, y) \in \mathcal{L}'_\mu \, .$$

Minimizing the right hand side with respect to λ, we find

$$L_\mu(\lambda, y) \geq -\frac{H(y)}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{H(y)} - \gamma - \frac{1}{2} + \frac{A}{H(y)\sqrt{\mu}}\right) \qquad \forall (\lambda, y) \in \mathcal{L}'_\mu \, .$$

Consequently, $L_\mu(\lambda, y) < a'_{\mu,\delta}$ implies that

$$(a-\delta)\exp(-\frac{2\sqrt{\mu}}{a-\delta} - \frac{A}{(a-\delta)\sqrt{\mu}}) < H(y)\exp\left(-\frac{2\sqrt{\mu}}{H(y)} + \frac{A}{H(y)\sqrt{\mu}}\right)$$

whence $H(y) > a - 2\delta$ for μ large enough.

Proof of (3.43). From (3.37) we deduce that for $(\lambda, y) \in \mathcal{L}'_\mu$

$$L_\mu(\lambda, y) \geq \frac{1}{\lambda}\left(2\sqrt{\mu} - H(y)(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma) - \frac{A}{\sqrt{\mu}}\right) \, .$$

Minimizing the right hand side with respect to λ, we find

$$L_\mu(\lambda, y) \geq -\frac{H(y)}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{H(y)} - \gamma - \frac{1}{2} + \frac{A}{H(y)\sqrt{\mu}}\right) \, .$$

The right hand side is, for large μ, a strictly decreasing function of $H(y)$, hence (3.43).

Remarks.

1. We have

$$\frac{\partial \widetilde{K}_\mu}{\partial \lambda} = \frac{\partial K_\mu}{\partial \lambda} + \frac{\partial K_\mu}{\partial v} \cdot \frac{\partial \bar{v}_\mu}{\partial \lambda} \, .$$

Proceeding as in [23], we find

$$\frac{\partial K_\mu}{\partial v} \cdot \frac{\partial \bar{v}_\mu}{\partial \lambda} = O\left(\frac{\mathrm{Log}\lambda}{\lambda^2}\|\bar{v}_\mu\|\right)$$

and using the same arguments as in Subsection 3.1, we obtain

$$\frac{\partial K_\mu}{\partial \lambda}(\lambda, y, \bar{v}_\mu) = \frac{\partial K_\mu}{\partial \lambda}(\lambda, y, 0) + O(\frac{1}{\lambda^2\sqrt{\mu}} + \frac{\mu}{\lambda^3}) .$$

Therefore, it follows from (3.10) and (C.47) that

(3.44)
$$\frac{\partial \widetilde{K}_\mu}{\partial \lambda}(\lambda, y) = 2\pi^{1/3}\left(-2\frac{\sqrt{\mu}}{\lambda^2} + \frac{H(y)}{\lambda^2}(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} - \gamma - \frac{1}{2})\right) + O\left(\frac{1}{\lambda^2\sqrt{\mu}} + \frac{\mu}{\lambda^3} + \frac{\sqrt{\mu}}{\lambda^3}\mathrm{Log}\frac{\lambda}{\sqrt{\mu}}\right) .$$

In the same way, proceeding as in [22, Appendix C], and using (C.48), we have

(3.45)
$$\frac{\partial^2 \widetilde{K}_\mu}{\partial \lambda^2}(\lambda, y) = 4\pi^{1/3}\left(2\frac{\sqrt{\mu}}{\lambda^3} - \frac{H(y)}{\lambda^3}(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} - \gamma - 1)\right) + O\left(\frac{1}{\lambda^3\sqrt{\mu}} + \frac{\mu}{\lambda^4} + \frac{\sqrt{\mu}}{\lambda^4}\mathrm{Log}\frac{\lambda}{\sqrt{\mu}}\right)$$

or

$$\frac{\partial^2 \widetilde{K}_\mu}{\partial \lambda^2}(\lambda, y) = -\frac{2}{\lambda}\frac{\partial \widetilde{K}}{\partial \lambda}(\lambda, y) + 2\pi^{1/3}\frac{H(y)}{\lambda^3} + O\left(\frac{1}{\lambda^3\sqrt{\mu}} + \frac{\mu}{\lambda^4} + \frac{\sqrt{\mu}}{\lambda^4}\mathrm{Log}\frac{\lambda}{\sqrt{\mu}}\right) . \tag{3.46}$$

From these formulae we deduce that for any $y \in H_c$, there is a unique $\lambda \in \left(e^{\frac{\sqrt{\mu}}{H}}, e^{\frac{3\sqrt{\mu}}{c}}\right)$ such that $\frac{\partial \widetilde{K}_\mu}{\partial \lambda}(\lambda, y) = 0$, which minimizes $\widetilde{K}_\mu(\lambda, y)$ with respect to λ.

2. Since $a''_{\delta,\mu} \leq L_\mu(\lambda_\mu, y_\mu) < a'_{\delta,\mu}$, we have

$$-\frac{2\sqrt{\mu}}{a-\delta} - \mathrm{Log}\frac{\sqrt{\mu}}{2(a-\delta)} - \gamma - \frac{1}{2} - \frac{A}{(a-\delta)\sqrt{\mu}}$$

$$\leq \mathrm{Log}(-L_\mu(\lambda_\mu, y_\mu)) \leq -\frac{2\sqrt{\mu}}{a+\delta} - \mathrm{Log}\frac{\sqrt{\mu}}{2(a+\delta)} - \gamma - \frac{1}{2} + \frac{A}{(a+\delta)\sqrt{\mu}}$$

whence, as δ may be chosen as small as desired as μ goes to infinity

$$\mathrm{Log}(-L_\mu(\lambda_\mu, y_\mu)) \sim -\frac{2\sqrt{\mu}}{a}$$

or equivalently

$$\mathrm{Log}\left(\frac{3\pi}{8} - \frac{1}{2\pi^{1/3}}I_\mu(u_\mu)\right) \sim -\frac{2\sqrt{\mu}}{a} \quad \text{as } \mu \text{ goes to infinity} \tag{3.47}$$

that is (1.6). Actually from (3.44) we derive

$$\mathrm{Log}\lambda_\mu = \frac{2\sqrt{\mu}}{H(y_\mu)} + \mathrm{Log}2\sqrt{\mu} + \gamma + \frac{1}{2} + O(\frac{1}{\sqrt{\mu}}) .$$

Therefore, using (3.35) and (3.37)

$$\mathrm{Log}\left(\frac{3\pi}{8} - \frac{1}{2\pi^{1/3}} I_\mu(u_\mu)\right) = -\frac{2\sqrt{\mu}}{H(y_\mu)} - \mathrm{Log}\frac{2\sqrt{\mu}}{H(y_\mu)} - \gamma - \frac{1}{2} + O(\frac{1}{\sqrt{\mu}})$$

hence (3.47), since $H(y_\mu)$ goes to a as μ goes to infinity. Moreover, from Proposition 3.2 we know that $\lambda_\mu \sim \frac{|u_\mu|^2_\infty}{\sqrt{3}}$ as $\mu \to \infty$. Therefore

$$\mathrm{Log}|u_\mu|_\infty \sim \frac{\sqrt{\mu}}{H(y_\mu)} \quad \text{as } \mu \to \infty$$

whence (1.7).

3.5 Proof of Theorem 1.4

We choose $A', A < A' \leq A+1$, such that

$$b'_\mu = -\frac{c}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{c} - \gamma - \frac{1}{2} + \frac{A'}{c\sqrt{\mu}}\right) < b_\mu \tag{3.48}$$

is not a critical value of L_μ. From the Ljusternik-Schnirelman theory [12] we derive that L_μ has at least as many distinct critical points in $L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu$ as $cat(L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu, L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu)$. On the other hand, we claim that for μ large enough

$$(L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu) \supset \mathcal{F}_\mu = \{(\lambda(y), y)/H(y) > c_\mu\} \qquad c_\mu = c(1 + \frac{A+1}{\mu}) . \tag{3.49}$$

Then

$$\mathrm{cat}(L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu, L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu) \geq \mathrm{cat}\left(\mathcal{F}_\mu, \mathcal{L}'_\mu\right)$$

whence

$$\mathrm{cat}(L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu, L_\mu^{b'_\mu} \cap \mathcal{L}'_\mu) \geq \mathrm{cat}(H_{c_\mu}, H_c) .$$

Indeed, if $\mathcal{F}_\mu$ may be covered by k closed sets F_i, $1 \leq i \leq k$, each one contractible in $\mathcal{L}'_\mu$, H_{c_μ} is covered by the k closed sets $G_i = \{y \in H_c / \exists \lambda \in [e^{\sqrt{\mu}/\bar{H}}, e^{3\sqrt{\mu}/c}], (\lambda, y) \in F_i\}, 1 \leq i \leq k$, each one contractible in H_c. As shown previously, each critical point of L_μ in $\mathcal{L}'_\mu$ provides us with a solution of (P). Therefore, for μ large enough, (P) has at least as many distinct solutions as $cat(H_{c_\mu}, H_c)$ - hence Theorem 1.4.

Proof of (3.49). From (3.37) and (3.40), we know that for any $y \in H_{c_\mu}$

$$L_\mu(\lambda(y), y) \leq -\frac{H(y)}{\lambda(y)} < -\frac{c_\mu}{2\sqrt{\mu}} \exp\left(-\frac{2\sqrt{\mu}}{c_\mu} - \gamma - \frac{1}{2} - \frac{A}{c_\mu\sqrt{\mu}}\right) .$$

Then

$$L_\mu(\lambda(y), y) < b'_\mu \frac{c_\mu}{c} \exp\left(2\sqrt{\mu}(\frac{1}{c} - \frac{1}{c_\mu}) - \frac{A}{c_\mu\sqrt{\mu}} - \frac{A'}{c\sqrt{\mu}}\right)$$

and the announced result follows, since

$$\frac{c_\mu}{c} \exp\left(2\sqrt{\mu}(\frac{1}{c} - \frac{1}{c_\mu}) - \frac{A}{c_\mu\sqrt{\mu}} - \frac{A'}{c\sqrt{\mu}}\right) = 1 + \frac{2 + A - A'}{c\sqrt{\mu}} + O(\frac{1}{\mu^{3/2}})$$

as μ goes to infinity, and $2 + A - A' \geq 1$.

APPENDIX

A Proof of Proposition 2.1

The equivalent of Proposition 2.1 has been proved in [23], following an argument by Bahri and Coron [8] with, instead of $V_{\lambda,y,\mu}, B_{\varepsilon,\mu}, \mathcal{V}_{\varepsilon,\mu}$ defined by (2.3), (2.1) and (2.2), $U_{\lambda,y}, B_\varepsilon, V_\varepsilon$ respectively, where $U_{\lambda,y}$ is given by (1.9) with $n = 3$ and

$$B_\varepsilon = \{(\alpha, \lambda, y) \in \mathbb{R} \times \mathbb{R}_+^* \times \partial\Omega / \frac{\bar{\alpha}}{2} < \alpha < 2\bar{\alpha}, \lambda > \frac{1}{\varepsilon}\}$$

and

$$V_\varepsilon = \{u \in H^1(\Omega) / \exists(\lambda, y) \in \partial\Omega, \lambda > \frac{1}{\varepsilon}, |\nabla(u - \bar{\alpha}U_{\lambda,y})|_2 < \varepsilon\} .$$

We are going to use the same kind of arguments as in [23], with the suitable modifications. We first state the following :

Lemma A.1 *Let (ε_m) be a sequence in $\mathbb{R}_+^*$ such that ε_m goes to zero as m goes to infinity, and $(\alpha_m, \lambda_m, y_m) \in B_{\varepsilon_m,\mu}$, $(\alpha'_m, \lambda'_m, y'_m) \in B_{\varepsilon_m,\mu}$ such that*

$$\text{(A.1)} \qquad |\nabla(\alpha_m V_{\lambda_m,y_m,\mu} - \alpha'_m V_{\lambda'_m y'_m,\mu})|_2 \to 0 \quad \text{as } m \to +\infty .$$

Then

$$\alpha_m - \alpha'_m \to 0 \quad \frac{\lambda_m}{\lambda'_m} \to 1 \quad \lambda_m \lambda'_m |y_m - y'_m|^2 \to 0 \quad \text{as } m \to +\infty .$$

Proof. From the integral estimates of Appendix C - see (C.42) (C.43) - we know that

$$\text{(A.2)} \qquad \int_\Omega |\nabla U_{\lambda,y}|^2 = \frac{3\pi^2}{8} + o(1) \quad \int_\Omega |\nabla \varphi_{\lambda,y,\mu}|^2 = o(1)$$

as $\lambda \to +\infty$, $\frac{\sqrt{\mu}}{\lambda} \to 0$. As λ_m, λ'_m are larger than $\frac{\sqrt{\mu}}{\varepsilon_m}$, with ε_m going to zero, (A.1) implies that

$$|\nabla(\alpha_m U_{\lambda_m,y_m} - \alpha'_m U_{\lambda'_m,y'_m})|_2 \to 0 \qquad \text{as } m \to +\infty .$$

Then, the same arguments as in [23] [8] apply and the conclusion follows.

The next result is :

Lemma A.2 *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon, 0 < \varepsilon < \varepsilon_0$, and any $u \in \mathcal{V}_{\varepsilon,\mu}$*

$$\inf_{(\alpha,\lambda,y)\in B_{4\varepsilon,\mu}} |\nabla(u - \alpha V_{\lambda,y,\mu})|_2$$

is achieved for some $(\alpha, \lambda, y) \in B_{2\varepsilon,\mu}$, and is not achieved in $B_{4\varepsilon,\mu} \setminus B_{2\varepsilon,\mu}$.

Proof. In a first step, we prove that the infimum is achieved in $\overline{B_{4\varepsilon,\mu}}$, and in a second step that it cannot be achieved in $\overline{B_{4\varepsilon,\mu}} \setminus B_{2\varepsilon,\mu}$, for ε small enough.

Step 1. Let $(\alpha^k, \lambda^k, y^k)$ be a minimizing sequence in $B_{4\varepsilon,\mu}$. Up to a subsequence, we may assume

$$\alpha^k \to \tilde{\alpha} \qquad \lambda^k \to \tilde{\lambda} \qquad y^k \to \tilde{y} \quad \text{as } m \to +\infty$$

with $\frac{\bar{\alpha}}{2} \leq \tilde{\alpha} \leq 2\bar{\alpha}$, $\frac{\sqrt{\mu}}{4\varepsilon} \leq \tilde{\lambda} \leq +\infty$, $\tilde{y} \in \partial\Omega$. The only thing to prove is that $\tilde{\lambda} < +\infty$, provided that ε is small enough. Arguing by contradiction, let us assume the existence of a sequence (ε_m) in $\mathbb{R}_+^*$, ε_m going to zero as m goes to infinity, a sequence (u_m) in $H^1(\Omega)$ with $u_m \in \mathcal{V}_{\varepsilon_m,\mu}$, and a sequence of minimizing sequences $(\alpha_m^k, \lambda_m^k, y_m^k)_k$ in $B_{4\varepsilon_m,\mu}$ such that for any m, λ_m^k goes to infinity as k goes to infinity. $u_m \in \mathcal{V}_{\varepsilon_m,\mu}$ means the existence of $\lambda_m > \frac{\sqrt{\mu}}{\varepsilon_m}$, $y_m \in \partial\Omega$ and $y_m \in H^1(\Omega)$ such that

$$u_m = \bar{\alpha} V_{\lambda_m,y_m,\mu} + v_m \qquad |\nabla v_m|_2 < \varepsilon_m \ .$$

$(\bar{\alpha}, \lambda_m, y_m) \in B_{\varepsilon_m,\mu} \subset B_{4\varepsilon_m,\mu}$. Therefore, the infinimum that we are interested in on $B_{4\varepsilon_m\mu}$ is strictly less than ε_m, and $(\alpha_m^k, \lambda_m^k, y_m^k)_k$ being a minimizing sequence, for k large enough, say $k \geq K(m)$, we have

$$|\nabla \left(\bar{\alpha} V_{\lambda_m,y_m,\mu} + v_m - \alpha_m^k V_{\lambda_m^k,y_m^k,\mu}\right)|_2 < \varepsilon_m$$

whence

$$|\nabla \left(\bar{\alpha} V_{\lambda_m,y_m,\mu} - \alpha_m^k V_{\lambda^k,y_m^k,\mu}\right)|_2 < 2\varepsilon_m \ .$$

From Lemma A.1 we infer the existence of M such that, for any $m \geq M$ and any $k \geq K(m)$

$$\frac{\lambda_m}{\lambda_m^k} \geq \frac{1}{2}$$

a contradiction with λ_m^k going to infinity as k goes to infinity. Hence the first step.

Step 2. We still argue by contradiction. Let us assume the existence of a sequence (ε_m) in $\mathbb{R}_+^*$ converging to zero, and a sequence (u_m) in $H^1(\Omega)$, $u_m \in \mathcal{V}_{\varepsilon_m,\mu}$, such that the infinimum of $|\nabla(u_m - \alpha V_{\lambda,y,\mu})|_2$, for $(\alpha, \lambda, y) \in B_{4\varepsilon_m,\mu}$, is achieved in $\overline{B_{4\varepsilon_m,\mu}} \setminus B_{2\varepsilon_m,\mu}$. Consequently, there exist $\lambda_m > \frac{\sqrt{\mu}}{\varepsilon_m}$ and $y_m \in \partial\Omega$ such that

$$|\nabla(u_m - \bar{\alpha} V_{\lambda_m,y_m,\mu})|_2 < \varepsilon_m$$

and, since $(\bar{\alpha}, \lambda_m, y_m) \in B_{4\varepsilon_m,\mu}$, some $(\alpha'_m, \lambda'_m, y'_m) \in \overline{B_{4\varepsilon_m,\mu}} \setminus B_{2\varepsilon_m,\mu}$ such that

$$|\nabla \left(u_m - \alpha'_m V_{\lambda'_m, y'_m, \mu}\right)|_2 < \varepsilon_m \, .$$

Therefore

$$|\nabla \left(\bar{\alpha} V_{\lambda_m, y_m, \mu} - \alpha'_m V_{\lambda'_m, y'_m, \mu}\right)|_2 < 2\varepsilon_m$$

and Lemma A.1 implies that λ_m/λ'_m goes to 1 as m goes infinity. As $\lambda_m > \frac{\sqrt{\mu}}{\varepsilon_m}$ and $\lambda'_m \leq \frac{\sqrt{\mu}}{2\varepsilon_m}$, this is a contradiction, hence the second step and Lemma A.2.

Proof of Proposition 2.1 completed. Once again, we argue by contradiction. If Proposition 2.1 is false, we derive from Lemma A.2 the existence of a sequence (ε_m) in $\mathbb{R}^*_+$, converging to zero, a sequence (u_m) in $H^1(\Omega)$, $u_m \in \mathcal{V}_{\varepsilon_m,\mu}$, and sequences $(\alpha_m, \lambda_m, y_m)$ and $(\alpha'_m, \lambda'_m, y'_m)$ in $\mathbb{R} \times \mathbb{R}^*_+ \times \partial\Omega$ such that for any m, $(\alpha_m, \lambda_m, y_m)$ and $(\alpha'_m, \lambda'_m, y'_m)$ are in $B_{2\varepsilon_m,\mu}$, $(\alpha_m, \lambda_m, y_m) \neq (\alpha'_m, \lambda'_m, y'_m)$, and

$$\inf_{(\alpha,\lambda,y)\in B_{4\varepsilon_m,\mu}} |\nabla(u_m - \alpha V_{\lambda,y,\mu})|_2 = |\nabla(u_m - \alpha_m V_{\lambda_m,y_m,\mu})|_2 = |\nabla(u_m - \alpha'_m V_{\lambda'_m,y'_m,\mu})|_2 \, .$$

We may write

$$u_m = \alpha_m V_{\lambda_m,y_m,\mu} + v_m = \alpha'_m V_{\lambda'_m,y'_m,\mu} + v'_m \tag{A.3}$$

with $v_m \in E_{\lambda_m,y_m,\mu}$, $v'_m \in E_{\lambda'_m,y'_m,\mu}$, and, since $u_m \in \mathcal{V}_{\varepsilon_m,\mu}$, $|\nabla v_m|_2 < \varepsilon_m$ and $|\nabla v'_m|_2 < \varepsilon_m$. We set

$$a_m = \alpha_m - \alpha'_m \qquad b_m = \frac{\lambda'_m}{\lambda_m} - 1 \qquad c_m = \lambda'_m(y_m - y'_m) \, .$$

Lemma A.1 shows that a_m, b_m, c_m go to zero as m goes to infinity. We are going to prove that for m large enough, $a_m = 0, b_m = 0, c_m = 0$ - a contradiction with $(\alpha_m, \lambda_m, y_m) \neq (\alpha'_m, \lambda'_m, y'_m)$, hence the desired conclusion. Omitting the index m, and V denoting $V_{\lambda,y,\mu}$, V' denoting $V_{\lambda',y',\mu}$, (A.3) implies

$$\int_\Omega \nabla(\alpha V - \alpha' V).\nabla V = \int_\Omega \nabla(v' - v).\nabla V = \int_\Omega \nabla v'.\nabla(V - V') \tag{A.4}$$

since $v \in E_{\lambda,y,\mu}$, $v' \in E_{\lambda',y',\mu}$, whence, as m goes to infinity

$$\int_\Omega \nabla(\alpha V - \alpha' V').\nabla V = o\left(\left(\int_\Omega |\nabla(V - V')|^2\right)^{1/2}\right) \, . \tag{A.5}$$

However, we may also write

$$\int_\Omega \nabla(\alpha V - \alpha' V').\nabla V = a \int_\Omega |\nabla V|^2 + \alpha' \int_\Omega \nabla(V - V').\nabla V \tag{A.6}$$

We claim that

$$\int_\Omega \nabla(V - V').\nabla V = o(|b| + |c|) \, . \tag{A.7}$$

Then, we have also

$$\int_\Omega |\nabla(V-V')|^2 = o(|b|+|c|)$$

and from (A.5), (A.6) and (A.2) we deduce that

$$a = o(|b|+|c|)\ .$$

Treating in the same way the quantities

$$\int_\Omega \nabla(\alpha V - \alpha' V').\nabla\frac{\partial V}{\partial\lambda}\ ,\ \int_\Omega \nabla(\alpha V-\alpha' V').\nabla\frac{\partial V}{\partial\tau_i}\quad i=1,2$$

we obtain

$$b = o(|a|+|c|)\qquad c=o(|a|+|b|)$$

whence, for m large enough, $a_m = 0$, $b_m = 0$, $c_m = 0$, as announced.

It only remains to prove (A.7). In [23] it is proved that

$$\int_\Omega \nabla(U-U').\nabla U = o(|b|+|c|)\ .$$

Then, we have

$$\int_\Omega \nabla(V-V').\nabla V = -\int_\Omega \nabla(U-U').\nabla\varphi + \int_\Omega \nabla(\varphi-\varphi').\nabla V + o(|b|+|c|)\ .$$

with $\varphi = \varphi_{\lambda,y,\mu}$, $\varphi' = \varphi_{\lambda',y',\mu}$. Easy computations show - see [23] - that

$$\text{(A.8)}\qquad |\nabla(U-U')| = O\left((|b|+|c|)|\nabla U| + |c|\frac{\lambda^{3/2}}{(1+\lambda^2|x-y|^2)^{3/2}}\right)\ .$$

On the other hand

$$\text{(A.9)}\qquad \nabla\varphi = \frac{1-e^{-\sqrt{\mu}|x-y|}-\sqrt{\mu}|x-y|e^{-\sqrt{\mu}(x-y)}}{\lambda^{1/2}|x-y|^3}(x-y)$$

so that

$$\text{(A.10)}\qquad |\nabla\varphi| = O\left(\inf\left(\frac{\mu}{\lambda^{1/2}},\frac{1}{\lambda^{1/2}|x-y|^2}\right)\right)\quad \text{uniformly in } \mathbb{R}^3\ .$$

Using (A.8) and (A.10), straightforward computations yield

$$\int_\Omega \nabla(U-U').\nabla\varphi = O\left((|b|+|c|)\frac{\sqrt{\mu}}{\lambda}\right)$$

so that

$$\text{(A.11)}\qquad \int_\Omega \nabla(U-U').\nabla\varphi = o(|b|+|c|)\ .$$

The last integral to estimate is $\int_\Omega \nabla(\varphi-\varphi').\nabla V$. Because of (A.10), we have

$$\int_{\substack{\Omega\\ |x-y|\le\frac{|c|}{\sqrt{\mu}}}} \nabla(\varphi-\varphi').\nabla V \;= O\left(\int_{|x|\le\frac{|c|}{\sqrt{\mu}}} \tfrac{\mu}{\lambda^{1/2}}\left(\tfrac{\lambda^{5/2}|x|}{(1+\lambda^2|x|^2)^{3/2}}+\tfrac{\mu}{\lambda^{1/2}}\right)dx\right)$$

$$= O\left(\tfrac{\sqrt{\mu}}{\lambda}|c|^3+\tfrac{\mu}{\lambda^2}\int_0^{\frac{\lambda|c|}{\sqrt{\mu}}}\tfrac{r^3dr}{(1+r^3)^{3/2}}\right)$$

$$= O\left(\tfrac{\sqrt{\mu}}{\lambda}|c|\right).$$

For $|x-y|\ge\frac{|c|}{\sqrt{\mu}}$, we have

$$|x-y'|=|x-y|+O(\frac{|c|}{\lambda})$$

and from (A.9) we derive

$$\nabla\varphi'=\frac{1-e^{-\sqrt{\mu}|x-y|}-\sqrt{\mu}|x-y|e^{-\sqrt{\mu}|x-y|}+O(\frac{\sqrt{\mu}}{\lambda}|c|)}{\lambda^{1/2}|x-y|^3}$$
$$.(1+O(|b|+\frac{|c|}{\lambda|x-y|}))(x-y+O(\frac{|c|}{\lambda}))\ .$$

Consequently, noticing that $1-e^{-\sqrt{\mu}|x-y|}-\sqrt{\mu}|x-y|e^{-\sqrt{\mu}|x-y|}=O(\mu|x-y|^2)$, we may write for $|x-y|\le\frac{|c|}{\sqrt{\mu}}$

$$\nabla(\varphi-\varphi')=O\left(\frac{1}{\lambda^{1/2}|x-y|^3}\left(\frac{\sqrt{\mu}}{\lambda}|c||x-y|+\mu|b||x-y|^3+\frac{\mu}{\lambda}|c||x-y|^2\right)\right)$$

and noticing that $|1-e^{-\sqrt{\mu}|x-y|}-\sqrt{\mu}|x-y|e^{-\sqrt{\mu}|x-y|}|\le 1$, we have for $|x-y|\ge\frac{|c|}{\sqrt{\mu}}$

$$\nabla(\varphi-\varphi')=O\left(\frac{1}{\lambda^{1/2}|x-y|^3}\left(\frac{\sqrt{\mu}}{\lambda}|c||x-y|+|b||x-y|+\frac{|c|}{\lambda}\right)\right)\ .$$

Therefore, using also (A.10), we have

$$\int_{\substack{\Omega\\ \frac{|c|}{\sqrt{\mu}}\le|x-y|\le\frac{1}{\sqrt{\mu}}}} \nabla(\varphi-\varphi').\nabla V$$
$$=\ O\left(\int_{\frac{|c|}{\sqrt{\mu}}\le|x|\le\frac{1}{\sqrt{\mu}}}\frac{1}{\lambda^{1/2}|x|^3}\left(\frac{\sqrt{\mu}}{\lambda}|c||x|+\mu|b||x|^3+\frac{\mu}{\lambda}|c||x|^2\right)\right.$$
$$\left.\cdot\left(\frac{\lambda^{5/2}|x|}{(1+\lambda^2|x|^2)^{5/2}}+\frac{\mu}{\lambda^{1/2}}\right)dx\right)$$
$$=\ O\left(\frac{1}{\lambda^{1/2}}\int_{\lambda|c|/\sqrt{\mu}}^{\lambda/\sqrt{\mu}}\left(\frac{\sqrt{\mu}}{\lambda^2}|c|+\frac{\mu}{\lambda^3}|b|r^2+\frac{\mu}{\lambda^2}|c|r\right)\left(\frac{\lambda^{3/2}r}{(1+r^2)^{3/2}}+\frac{\mu}{\lambda^{1/2}}\right)dr\right)$$
$$=\ O\left(\frac{\sqrt{\mu}}{\lambda}(|b|+|c|)\right)$$

and

$$\int_{\substack{\Omega \\ |x-y|\geq \frac{1}{\sqrt{\mu}}}} \nabla(\varphi-\varphi').\nabla V$$

$$\begin{aligned}
&= O\left(\int_{|x|\geq \frac{1}{\sqrt{\mu}}} \frac{1}{\lambda^{1/2}|x|^3}\left(\frac{\sqrt{\mu}}{\lambda}|c||x| + |b||x| + \frac{|c|}{\lambda}\right)\left(\frac{\lambda^{5/2}|x|}{(1+\lambda^2|x|^2)^{3/2}} + \frac{1}{\lambda^{1/2}|x|^2}\right) dx\right) \\
&= O\left(\frac{1}{\lambda^{1/2}}\int_{\lambda/\sqrt{\mu}}^{+\infty}\left(\frac{\sqrt{\mu}|c|}{\lambda^2}r + \frac{|b|}{\lambda}r + \frac{|c|}{\lambda}\right)\frac{\lambda^{3/2}}{r^2}\frac{dr}{r}\right) \\
&= O\left(\frac{\sqrt{\mu}}{\lambda}|b| + \frac{\mu}{\lambda^2}|c|\right) .
\end{aligned}$$

Collecting these results, we obtain

$$\int_\Omega \nabla(\varphi-\varphi').\nabla V = o(|b|+|c|)$$

and the proof of (A.7) is complete.

B The coercivity of Q.

Our aim is this section is to prove Lemma 3.2. From (3.2) we know that for $v \in E_{\lambda,y,\mu}$

$$\mu\int_\Omega Vv = O\left(\frac{\mu^{1/4}}{\lambda^{1/2}}\|v\|\right)$$

and from (3.12) (3.13) (3.14) (3.15) (3.17) (3.18) (3.2)

$$\int_\Omega V^5 v = O\left(\frac{1}{\lambda^{1/2}\mu^{1/4}}\|v\|\right) .$$

In view of the definition (3.5) of Q and taking account of (3.11), we obtain

$$Q(v) = \frac{2}{\pi^{2/3}}\left(\|v\|^2 - 15\int_\Omega V^4v^2\right) + o(\|v\|^2) \quad \text{as } \frac{\sqrt{\mu}}{\lambda}\to 0 .$$

We write

$$\int_\Omega V^4v^2 = \int_\Omega U^4v^2 + O\left(\int_\Omega U^3\varphi v^2 + \int_\Omega \varphi^4 v^2\right) .$$

On one hand

$$\int_\Omega U^3\varphi v^2 \leq C\left(\int_\Omega U^{9/2}\varphi^{3/2}\right)^{2/3}|v|^2_{H^1} = O\left(\frac{\sqrt{\mu}}{\lambda}|v|^2_{H^1}\right)$$

according to (C.50), and on the other hand

$$\int_\Omega \varphi^4 v^2 \leq C\left(\int_\Omega \varphi^6\right)^{2/3} |v|^2_{H^1} = O\left(\frac{\mu}{\lambda^2}|v|^2_{H^1}\right)$$

according to (C.12). Therefore

$$Q(v) = \frac{2}{\pi^{2/3}}\left(\|v\|^2 - 15\int_\Omega U^4 v^2\right) + o(\|v\|^2) \quad \text{as } \frac{\sqrt{\mu}}{\lambda} \to 0$$

and we are led to prove the coercivity of the quadratic form

$$\widetilde{Q}(v) = \|v\|^2 - 15\int_\Omega U^4 v^2 + o(\|v\|^2) \qquad v \in E_{\lambda,y,\mu} .$$

Adapting to the Neumann case the arguments of [7] [21], it is proved in [4, Lemma 3.4] that

$$\text{(B.1)} \qquad \int_\Omega |\nabla v|^2 + \omega\int_\Omega v^2 - 15\int_\Omega U^4 v^2 \geq \rho\left(\int_\Omega |\nabla v|^2 + \omega\int_\Omega v^2\right) \quad \forall v \in E_{\lambda,y}$$

provided that $\frac{\sqrt{\omega}}{\lambda}$ is small enough, with ρ a strictly positive constant and

$$E_{\lambda,y} = \left\{v \in H^1(\Omega) / \int_\Omega \nabla v.\nabla U = \int_\Omega \nabla v.\nabla\frac{\partial U}{\partial\lambda} = \int_\Omega \nabla v.\nabla\frac{\partial U}{\partial\tau_i} = 0, i = 1,2\right\} .$$

Let us deduce from this result the coercivity of $\widetilde{Q}$, provided that $\frac{\sqrt{\mu}}{\lambda}$ is small enough. Let $v \in E_{\lambda,y,\mu}$. We write

$$\text{(B.2)} \qquad v = w + z \qquad w \in E_{\lambda,y} \qquad z \in \mathrm{Vect}\left(U, \frac{\partial U}{\partial\lambda}, \frac{\partial U}{\partial\tau_i}, i = 1,2\right)$$

that is

$$\text{(B.3)} \qquad z = aU + b\frac{\partial U}{\partial\lambda} + \sum_{i=1,2} c_i\frac{\partial U}{\partial\tau_i} \qquad a, b, c_i \in \mathbb{R} .$$

We multiply the gradient of (B.2) by $\nabla U, \nabla\frac{\partial U}{\partial\lambda}, \nabla\frac{\partial U}{\partial\tau_i}$ respectively, and integrate over Ω. On the left hand side, we find

$$\int_\Omega \nabla v.\nabla U = \int_\Omega \nabla v.\nabla\varphi = O\left(\frac{\mu^{1/4}}{\sqrt{\lambda}}|\nabla v|_2\right)$$

since $v \in E_{\lambda,y,\mu}$ and using (C.43). In the same way

$$\int_\Omega \nabla v.\nabla\frac{\partial U}{\partial\lambda} = \int_\Omega \nabla v.\nabla\frac{\partial\varphi}{\partial\lambda} = O\left(\frac{\mu^{1/4}}{\lambda^{3/2}}|\nabla v|_2\right)$$

and

$$\int_\Omega \nabla v.\nabla \frac{\partial U}{\partial \tau_i} = \int_\Omega \nabla v.\nabla \frac{\partial \varphi}{\partial \tau_i} = O\left(\frac{\mu^{3/4}}{\sqrt{\lambda}}|\nabla v|_2\right)$$

since $|\nabla \frac{\partial \varphi}{\partial \lambda}|_2 = O(\frac{\mu^{1/4}}{\lambda^{3/2}})$ and $|\nabla \frac{\partial \varphi}{\partial \tau_i}| = O(\frac{\mu^{3/4}}{\sqrt{\lambda}})$, as easy computations show. Concerning the right hand side, we have - see [23, Appendix D]

$$\text{(B.4)} \qquad \begin{cases} \int_\Omega |\nabla U|^2 = \frac{3\pi^2}{8} + O(\frac{\mathrm{Log}\lambda}{\lambda^2}) & \int_\Omega \nabla U.\nabla \frac{\partial U}{\partial \lambda} = O(\frac{\mathrm{Log}\lambda}{\lambda^2}) \\ \int_\Omega |\nabla \frac{\partial U}{\partial \lambda}|^2 = \frac{15\pi^2}{128\lambda^2} + O(\frac{\mathrm{Log}\lambda}{\lambda^3}) & \int_\Omega \nabla U.\nabla \frac{\partial U}{\partial \tau_i} = O(\frac{\mathrm{Log}\lambda}{\lambda}) \\ \int_\Omega \nabla \frac{\partial U}{\partial \tau_j}.\nabla \frac{\partial U}{\partial \tau_j} = \frac{15\pi^2}{128}\lambda^2\delta_{ij} + O(\lambda) & \int_\Omega \nabla U.\nabla \frac{\partial U}{\partial \lambda} = O(\frac{\mathrm{Log}\lambda}{\lambda^2}) \end{cases}$$

Thus, a, b, c_1, c_2 satisfy a 4×4 linear system which is nearly diagonal and whose inversion provides us with the estimates

$$\text{(B.5)} \qquad a = O\left(\frac{\mu^{1/4}}{\sqrt{\lambda}}|\nabla v|_2\right) \quad b = O\left(\mu^{1/4}\sqrt{\lambda}|\nabla v|_2\right) \quad c_i = O\left(\frac{\mu^{3/4}}{\lambda^{5/2}}|\nabla v|_2\right) .$$

From (B.3) (B.4) (B.5) we deduce that

$$|\nabla z|_2 = O\left(\frac{\mu^{1/4}}{\sqrt{\lambda}}|\nabla v|_2\right) .$$

Straightforward computations show that

$$\text{(B.6)} \qquad \int_\Omega U^2 = O(\frac{1}{\lambda}) \qquad \int_\Omega (\frac{\partial U}{\partial \lambda})^2 = O(\frac{1}{\lambda^2}) \qquad \int_\Omega (\frac{\partial U}{\partial \tau_i})^2 = O(1)$$

whence, using (B.3) and (B.5)

$$|z|_2 = O\left(\frac{\mu^{1/4}}{\lambda}|\nabla v|_2\right) .$$

For $h \in H^1(\Omega)$, we denote by $||h||_{\sqrt{\mu}}$ the quantity

$$||h||_{\sqrt{\mu}} = \left(\int_\Omega |\nabla h|^2 + \sqrt{\mu}\int_\Omega h^2\right)^{1/2} .$$

Then, we have

$$||z||_{\sqrt{\mu}} = o(||v||_{\sqrt{\mu}}) \qquad ||v||_{\sqrt{\mu}} = ||w||_{\sqrt{\mu}}(1 + o(1))$$

as $\frac{\sqrt{\mu}}{\lambda}$ goes to zero, and

$$\begin{aligned} ||v||^2_{\sqrt{\mu}} - 15\int_\Omega U^4 v^2 &= ||w||^2_{\sqrt{\mu}} - 15\int_\Omega U^4 w^2 + O(||w||_{\sqrt{\mu}}||z||_{\sqrt{\mu}} + ||z||^2_{\sqrt{\mu}}) \\ &\geq \rho||w||^2_{\sqrt{\mu}} + o(||w||^2_{\sqrt{\mu}}) \\ &\geq \rho/2||v||^2_{\sqrt{\mu}} \end{aligned}$$

provided that μ is large enough and $\frac{\sqrt{\mu}}{\lambda}$ is small enough, because of (B.1) with $\omega = \sqrt{\mu}$. Finally, we write

$$\begin{aligned}\widetilde{Q}(v) &= \|v\|^2_{\sqrt{\mu}} - 15\int_\Omega U^4 v^2 + (\mu - \sqrt{\mu})\int_\Omega v^2 \\ &\geq \rho/2\|v\|^2_{\sqrt{\mu}} + (\mu - \sqrt{\mu})\int_\Omega v^2 \\ &\geq \rho'\|v\|^2\end{aligned}$$

with $\rho' = \min(\frac{\rho}{2}, 1 - \frac{1-\rho/2}{\sqrt{\mu}}) > 0$ for μ large enough, and Lemma 2.2 is proved.

C Estimates

In this last section we collect the different integral estimates which occur in the proof of the theorems. Assuming that $y \in \partial\Omega$, μ goes to infinity and $\frac{\sqrt{\mu}}{\lambda}$ goes to zero, we establish the following asymptotic expansions :

$$\text{(C.1)} \qquad \int_\Omega |\nabla V_{\lambda,y,\mu}|^2 = \frac{3\pi^2}{8} - 3\pi\frac{\sqrt{\mu}}{\lambda} - \pi\frac{H(y)}{\lambda}\left(\text{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma\right) + O(\frac{1}{\lambda\sqrt{\mu}})$$

$$\text{(C.2)} \qquad \int_\Omega V^2_{\lambda,y,\mu} = \frac{\pi}{\lambda\sqrt{\mu}} - \frac{\pi}{4}\frac{H(y)}{\lambda\mu} + O(\frac{1}{\lambda\mu^{3/2}})$$

$$\text{(C.3)} \qquad \int_\Omega V^6_{\lambda,y,\mu} = \frac{\pi^2}{8} - 4\pi\frac{\sqrt{\mu}}{\lambda} - \frac{\pi}{4}\frac{H(y)}{\lambda} + O(\frac{\sqrt{\mu}}{\lambda^2})$$

where γ denotes the Euler constant. From this we deduce

(C.4)

$$K_\mu(\lambda, y, 0) = 2\pi^{1/3}\left(\frac{3\pi}{8} + 2\frac{\sqrt{\mu}}{\lambda} - \frac{H(y)}{\lambda}(\text{Log}\frac{\lambda}{2\sqrt{\mu}} + \frac{1}{2} - \gamma)\right) + O\left(\frac{1}{\lambda\sqrt{\mu}} + \frac{\mu}{\lambda^2} + \frac{\sqrt{\mu}}{\lambda^2}\text{Log}\frac{\lambda}{\sqrt{\mu}}\right) .$$

Proof. Up to a translation and a rotation of the coordinates in $\mathbb{R}^3$, we can assume that $y = 0$, and that for $R > 0$ small enough

$$\text{(C.5)} \qquad \Omega \cap B(0, R) = \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}/|x| < R, x_3 > f(x')\}$$

with

$$\text{(C.6)} \qquad f(x') = f_1 x_1^2 + f_2 x_2^2 + O(|x'|^2) \qquad |x'| < R\ .$$

Note that we have

$$\text{(C.7)} \qquad H(0) = f_1 + f_2$$

and we choose some $R' > 0$ such that

$$\Omega \subset B(0, R') \,. \tag{C.8}$$

Let us begin with the proof of (C.3). It follows from the definition (2.3) of $V_{\lambda,0,\mu} = V$ that

$$V(x) = O\left(\frac{1}{\lambda^{5/2}|x|} + \frac{e^{-\sqrt{\mu}|x|}}{\lambda^{1/2}|x|}\right) \text{ uniformly for } |x| > T > 0 \,. \tag{C.9}$$

Therefore

$$\int_{\Omega \cap B_R^c} V^6 = O\left(\frac{1}{\lambda^{15}} + \frac{e^{-6R\sqrt{\mu}}}{\lambda^3\sqrt{\mu}}\right) \,. \tag{C.10}$$

For $x \in B_R$, we write (with $U = U_{\lambda,0}$, $\varphi = \varphi_{\lambda,0,\mu}$)

$$V^6 = U^6 - 6U^5\varphi + O(U^4\varphi^2 + \varphi^6) \,. \tag{C.11}$$

The last terms are easy to estimate. Namely, we have

$$\int_\Omega \varphi^6 \leq \frac{4\pi}{\lambda^3}\int_o^{R'} \frac{(1 - e^{-\sqrt{\mu}r})^6}{r^4} dr = \frac{4\pi\mu^{3/2}}{\lambda^3}\int_0^{\sqrt{\mu}R'} \frac{(1 - e^{-t})^6}{t^4} dt = O\left(\frac{\mu^{3/2}}{\lambda^3}\right) \tag{C.12}$$

and

$$\int_\Omega U^4\varphi^2 \leq 4\pi \int_0^{R'} \frac{\lambda(1 - e^{-\sqrt{\mu}r})^2}{(1 + \lambda^2 r^2)^2} dr \leq \frac{4\pi\mu}{\lambda^2}\int_0^{\lambda R'} \frac{t^2}{(1 + t^2)} dt = O\left(\frac{\mu}{\lambda^2}\right) \tag{C.13}$$

so that

$$\int_\Omega (U^4\varphi^2 + \varphi^6) = O\left(\frac{\mu}{\lambda^2}\right) \,. \tag{C.14}$$

Concerning the remaining terms, we note that for W a function defined in $\mathbb{R}^3$, we have

$$\int_{\Omega \cap B_R} W = \int_{B_R^+} W - \int_{\omega'} W + \int_{\omega''} W \tag{C.15}$$

with

$$\begin{cases} B_R^+ = & \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}, |x| < R, x_3 > 0\} \\ \omega' = & \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}, |x| < R, 0 < x_3 < f(x')\} \\ \omega'' = & \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}, |x| < R, f(x') < x_3 < 0\} \,. \end{cases} \tag{C.16}$$

Let $W = U^6$. We compute

$$\int_{B_R^+} U^6 = \frac{1}{2}\int_{B_R} U^6 = 2\pi \int_0^{\lambda R} \frac{r^2}{(1 + r^2)^3} dr = 2\pi \int_0^{+\infty} \frac{r^2}{(1 + r^2)^3} dr + O(\frac{1}{\lambda^3})$$

that is

$$\int_{B_R^+} U^6 = \frac{\pi^2}{8} + O(\frac{1}{\lambda^3}) \ . \tag{C.17}$$

Let a be a strictly positive constant such that

$$\Delta_a = \{x' \in \mathbb{R}^2 / |x'| < a\} \tag{C.18}$$

and we denote by L_a the cylinder

$$L_a = \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} / x' \in \Delta_a\} \ . \tag{C.19}$$

Then we may write

$$\int_{\omega'} U^6 - \int_{\omega''} U^6 = \int_{\omega' \cap L_a} U^6 - \int_{\omega'' \cap L_a} U^6 + O(\frac{1}{\lambda^3}) \tag{C.20}$$

since, outside L_a, $U = O(\lambda^{-1/2})$, and we have

$$\begin{aligned}
\int_{\omega' \cap L_a} U^6 - \int_{\omega'' \cap L_a} U^6 &= \int_{\Delta_a} \left(\int_0^{f(x')} \frac{\lambda^3}{(1+\lambda^2|x|^2)^3} dx_3 \right) dx' \\
&= \int_{\Delta_a} \frac{\lambda^2}{(1+\lambda^2|x'|^2)^{5/2}} \left(\int_0^{\frac{\lambda f(x')}{(1+\lambda^2|x|^2)^{1/2}}} \frac{dz}{(1+z^2)^3} \right) dx' \\
&= \int_{\Delta_a} \left(\frac{\lambda^3 f(x')}{(1+\lambda^2|x'|^2)^3} + O\left(\frac{\lambda^5 |f(x')|^3}{(1+\lambda^2|x'|^2)^4} \right) \right) dx'
\end{aligned}$$

noticing that

$$\int_0^{\alpha} \frac{dz}{(1+z^2)^{\beta}} = \alpha + O(\alpha^3) \qquad \text{for } \alpha > 0, \beta > \frac{1}{2} \ . \tag{C.21}$$

In view of (C.6) (C.7) we obtain

$$\begin{aligned}
\int_{\omega' \cap L_a} U^6 - \int_{\omega'' \cap L_a} U^6 &= \lambda^3 \sum_{i=1,2} f_i \int_{\Delta_a} \frac{x_i^2 dx'}{(1+\lambda^2|x'|^2)^3} + O\left(\int_{\Delta_a} \frac{\lambda^3 |x'|^3}{(1+\lambda^2|x'|^2)^3} dx' \right) \\
&= \frac{H(0)}{2\lambda} \int_{\lambda\Delta_a} \frac{|x'|^2 dx'}{(1+|x'|^2)^3} + O\left(\frac{1}{\lambda^2} \int_{\lambda\Delta_a} \frac{|x'|^3 dx'}{(1+|x'|^2)^3} \right) \\
&= \frac{\pi H(0)}{\lambda} \int_0^{+\infty} \frac{r^3 dr}{(1+r^2)^3} + O(\frac{1}{\lambda^2})
\end{aligned}$$

so that finally

$$\int_{\omega' \cap L_a} U^6 - \int_{\omega'' \cap L_a} U^6 = \frac{\pi H(0)}{4\lambda} + O(\frac{1}{\lambda^2}) \ . \tag{C.22}$$

Lastly, in view of (C.11) and (C.15), with $W = U^5\varphi$, we compute

$$\int_{B_R^+} U^5\varphi = \frac{1}{2}\int_{B_R} U^5\varphi = 2\pi\lambda^2\int_0^R \frac{(1-e^{-\sqrt{\mu}r})r}{(1+\lambda^2r^2)^{5/2}}dr = 2\pi\int_0^{\lambda R}\frac{(1-e^{-\frac{\sqrt{\mu}}{\lambda}r})r}{(1+r^2)^{5/2}}dr\ .$$

Writing $1-e^{-\frac{\sqrt{\mu}}{\lambda}r} = \frac{\sqrt{\mu}}{\lambda}r + O(\frac{\mu}{\lambda^2}r^2)$, we find

$$\begin{aligned}\int_{B_R^+} U^5\varphi &= \frac{2\pi\sqrt{\mu}}{\lambda}\int_0^{\lambda R}\frac{r^2dr}{(1+r^2)^{5/2}} + O\left(\frac{\mu}{\lambda^2}\int_0^{\lambda R}\frac{r^3dr}{(1+r^2)^{5/2}}\right)\\ &= \frac{2\pi\sqrt{\mu}}{\lambda}\int_0^{+\infty}\frac{r^2dr}{(1+r^2)^{5/2}} + O(\frac{\mu}{\lambda^2})\end{aligned}$$

that is

$$\int_{B_R^+} U^5\varphi = \frac{2\pi\sqrt{\mu}}{3\lambda} + O(\frac{\mu}{\lambda^2})\ . \tag{C.23}$$

Outside L_a, $U^5\varphi = O(1/\lambda^3)$. Therefore

$$\int_{\omega'} U^5\varphi - \int_{\omega''} U^5\varphi = \int_{\omega'\cap L_a} U^5\varphi - \int_{\omega''\cap L_a} U^5\varphi + O(\frac{1}{\lambda^3}) \tag{C.24}$$

and we have, since $U^5\varphi \le \frac{\lambda^2\sqrt{\mu}}{(1+\lambda^2|x|^2)^{5/2}}$

$$\begin{aligned}\int_{\omega'\cap L_a} U^5\varphi - \int_{\omega''\cap L_a} U^5\varphi &= O\left(\lambda^2\sqrt{\mu}\int_{\Delta_a}\frac{|f(x')|}{(1+\lambda^2|x'|^2)^{5/2}}dx'\right)\\ &= O\left(\lambda^2\sqrt{\mu}\int_0^a\frac{r^3dr}{(1+\lambda^2r^2)^{5/2}}\right)\\ &= O(\frac{\sqrt{\mu}}{\lambda^2})\ .\end{aligned}$$

This result, together with (C.10) (C.11) (C.14) (C.15) (C.17) (C.22) (C.23) (C.24), completes the proof of (C.3).

We turn now to the proof of (C.2). According to (C.9), we know that

$$\int_{\Omega\cap B_R^c} V^2 = O\left(\frac{1}{\lambda^5} + \frac{e^{-2R\sqrt{\mu}}}{\lambda\sqrt{\mu}}\right) \tag{C.25}$$

On $\Omega\cap B_R$, we still use (C.15) with $W = V^2$, and we have

$$\begin{aligned}\int_{B_R^+} V^2 = \frac{1}{2}\int_{B_R} V^2 &= 2\pi\int_0^R\left(\frac{\lambda^{1/2}}{(1+\lambda^2r^2)^{1/2}} - \frac{1-e^{-\sqrt{\mu}r}}{\lambda^{1/2}r}\right)^2 r^2dr\\ &= \frac{2\pi}{\lambda^2}\int_0^{\lambda R}\left(\frac{1}{(1+r^2)^{1/2}} - \frac{1-e^{-\frac{\sqrt{\mu}}{\lambda}r}}{r}\right)^2 r^2dr\ .\end{aligned}$$

As a consequence, we may write

$$
\begin{aligned}
\int_{B_R^+} V^2 &= \frac{2\pi}{\lambda^2}\left(\int_1^{\lambda R}\left(\frac{e^{-\frac{\sqrt{\mu}}{\lambda}r}}{r} + O(\frac{1}{r^3})\right)^2 r^2 dr + O(1)\right)\\
&= \frac{2\pi}{\lambda^2}\left(\int_1^{\lambda R}\left(e^{-2\frac{\sqrt{\mu}}{\lambda}r} + O(\frac{e^{-\frac{\sqrt{\mu}}{\lambda}r}}{r^2} + \frac{1}{r^4})\right) dr + O(1)\right)\\
&= \frac{2\pi}{\lambda^2}\left(\frac{\lambda}{2\sqrt{\mu}}\left(e^{-2\frac{\sqrt{\mu}}{\lambda}} - e^{-2R\sqrt{\mu}}\right) + O(1)\right)
\end{aligned}
$$

whence

$$
\int_{B_R^+} V^2 = \frac{\pi}{\lambda\sqrt{\mu}} + O\left(\frac{1}{\lambda^2} + \frac{e^{-2R\sqrt{\mu}}}{\lambda\sqrt{\mu}}\right) . \tag{C.26}
$$

On the other hand

$$
\int_{\omega'} V^2 - \int_{\omega''} V^2 = \int_{\omega'\cap L_a} V^2 - \int_{\omega''\cap L_a} V^2 + O\left(\frac{1}{\lambda^5} + \frac{e^{-2a\sqrt{\mu}}}{\lambda\sqrt{\mu}}\right)
$$

because of (C.9). We notice that $V = O(\lambda^{1/2})$ for $x \in \mathbb{R}^3$ and

$$
V(x) = \frac{e^{-\sqrt{\mu}|x|}}{\lambda^{1/2}|x|} + O\left(\frac{1}{\lambda^{5/2}|x|^3}\right) \qquad \text{for } |x| > \frac{1}{\lambda} .
$$

Therefore

$$
\begin{aligned}
&\int_{\omega'\cap L_a} V^2 - \int_{\omega''\cap L_a} V^2\\
&= \int_{\frac{1}{\lambda}<|x'|<a}\left(\int_0^{f(x')}\left(\frac{e^{-\sqrt{\mu}|x|}}{\lambda^{1/2}|x|} + O\left(\frac{1}{\lambda^{5/2}|x|^3}\right)\right)^2 dx_3\right) dx' + O\left(\lambda\int_{|x'|\le\frac{1}{\lambda}} |f(x')|dx'\right)\\
&= \int_{\frac{1}{\lambda}<|x'|\le a}\left(\int_0^{f(x')}\left(\frac{e^{-2\sqrt{\mu}|x|}}{\lambda|x|^2} + O\left(\frac{e^{-\sqrt{\mu}|x|}}{\lambda^3|x|^4} + \frac{1}{\lambda^5|x|^6}\right)\right) dx_3\right) dx'\\
&\qquad + O\left(\lambda\int_{|x'|\le\frac{1}{\lambda}} |x'|^2 dx'\right)
\end{aligned}
$$

because of (C.6). Since $|x'| \le |x|$ and, using (C.6)

$$
|x| = |x'|(1 + O(|x'|^2)) \qquad e^{-\sqrt{\mu}|x|} = e^{-\sqrt{\mu}|x'|}(1 + O(\sqrt{\mu}|x'|^3)) \text{ for } x \in (\omega' \cup \omega'') \tag{C.27}
$$

we obtain

$$\int_{\omega'\cap L_a} V^2 - \int_{\omega''\cap L_a} V^2 = \int_{\frac{1}{\lambda}<|x'|<a} \left(\frac{e^{-2\sqrt{\mu}|x'|}}{\lambda |x'|^2} f(x') + O\left(\left(\frac{e^{-2\sqrt{\mu}|x'|}}{\lambda |x'|^2} \left(|x'|^2 + \sqrt{\mu}|x'|^3\right) + \frac{e^{-\sqrt{\mu}|x'|}}{\lambda^3 |x'|^4} + \frac{1}{\lambda^5 |x'|^6} \right) |f(x')| \right) \right) dx' + O(\frac{1}{\lambda^3})$$

$$= \frac{1}{\lambda} \sum_{i=1,2} f_i \int_{\frac{1}{\lambda}<|x'|<a} \frac{e^{-2\sqrt{\mu}|x'|} x_i^2}{|x'|^2} dx' + O\left(\frac{1}{\lambda} \int_{\frac{1}{\lambda}}^{a} \left(e^{-2\sqrt{\mu}r}(r^2 + \sqrt{\mu}r^4) + \frac{e^{-\sqrt{\mu}r}}{\lambda^2 r} + \frac{1}{\lambda^4 r^3} \right) dr + \frac{1}{\lambda^3} \right)$$

$$= \frac{H(0)}{2\lambda} \int_{\frac{1}{\lambda}<|x'|<a} e^{-2\sqrt{\mu}|x'|} dx' + O\left(\frac{1}{\lambda \mu^{3/2}} \int_{2\frac{\sqrt{\mu}}{\lambda}}^{2a\sqrt{\mu}} e^{-t}(t^2 + \frac{t^4}{\sqrt{\mu}})\, dt + \frac{1}{\lambda^3} \int_{\frac{\sqrt{\mu}}{\lambda}}^{a\sqrt{\mu}} \frac{e^{-t}}{t} dt + \frac{1}{\lambda^3} \right)$$

$$= \frac{\pi H(0)}{4\lambda\mu} \left(\int_0^{+\infty} e^{-r} r dr - \int_0^{2\frac{\sqrt{\mu}}{\lambda}} e^{-r} r dr - \int_{2a\sqrt{\mu}}^{+\infty} e^{-r} r dr \right) + O\left(\frac{1}{\lambda \mu^{3/2}} + \frac{\mathrm{Log}\lambda}{\lambda^3} \right)$$

$$= \frac{\pi H(0)}{4\lambda\mu} + O\left(\frac{1}{\lambda\mu^{3/2}} + \frac{e^{-2a\sqrt{\mu}}}{\lambda\sqrt{\mu}} \right)$$

hence, collecting this result with (C.25) (C.15) (C.26), we find (C.2).

We prove now (C.1). According to the definition (2.3) of V, we have

$$\frac{\partial V}{\partial x_i}(x) = -\frac{\lambda^{5/2} x_i}{(1+\lambda^2|x|^2)^{3/2}} + \frac{(1 - e^{-\sqrt{\mu}|x|} - \sqrt{\mu}|x| e^{-\sqrt{\mu}|x|}) x_i}{\lambda^{1/2}|x|^3} \tag{C.28}$$

and so

$$|\nabla V|^2 = \frac{\lambda^5 |x|^2}{(1+\lambda^2|x|^2)^3} - \frac{2\lambda^2 (1 - e^{-\sqrt{\mu}|x|} - \sqrt{\mu}|x| e^{-\sqrt{\mu}|x|})}{(1+\lambda^2|x|^2)^{3/2}|x|} + \frac{(1 - e^{-\sqrt{\mu}|x|} - \sqrt{\mu}|x| e^{-\sqrt{\mu}|x|})^2}{\lambda |x|^4}. \tag{C.29}$$

We have

$$|\nabla V|^2 = O\left(\frac{1}{\lambda^5 r^8} + \frac{\mu e^{-2\sqrt{\mu}r}}{\lambda r^2} \right) \quad \text{uniformly for } |x| > T > 0 \tag{C.30}$$

so that

$$\int_{\Omega \cap B_R^c} |\nabla V|^2 = O\left(\frac{1}{\lambda^5} + \frac{\sqrt{\mu} e^{-2R\sqrt{\mu}}}{\lambda} \right). \tag{C.31}$$

On $\Omega \cap B_R$, in view of (C.15) with $W = |\nabla V|^2$, and (C.29), we compute

$$
\begin{aligned}
\int_{B_R^+} \frac{\lambda^2|x|^2}{(1+\lambda^2|x|^2)^3}dx &= 2\pi \int_0^{\lambda R} \frac{r^4 dr}{(1+r^2)^3} \\
&= 2\pi \int_0^{+\infty} \frac{r^4 dr}{(1+r^2)^3} - 2\pi \int_{\lambda R}^{+\infty} \frac{dr}{1+r^2} + O\left(\int_{\lambda R}^{+\infty} \frac{dr}{(1+r^2)^2}\right) \\
&= \frac{3\pi^2}{8} - 2\pi Arctg \frac{1}{\lambda R} + O(\frac{1}{\lambda^3}) \\
&= \frac{3\pi^2}{8} - \frac{2\pi}{\lambda R} + O(\frac{1}{\lambda^3})
\end{aligned}
$$

$$
\begin{aligned}
&\int_{B_R^+} \frac{2\lambda^2(1-e^{-\sqrt{\mu}|x|} - \sqrt{\mu}|x|e^{-\sqrt{\mu}|x|})}{(1+\lambda^2|x|^2)^{3/2}|x|}dx = 2\pi \int_0^R \frac{1-e^{-\sqrt{\mu}r} - \sqrt{\mu}re^{-\sqrt{\mu}r}}{(1+\lambda^2 r^2)^{3/2}} 2\lambda^2 r dr \\
&= 2\pi \left(\frac{-2(1-e^{-R\sqrt{\mu}} - R\sqrt{\mu}e^{-R\sqrt{\mu}})}{(1+\lambda^2 R^2)^{1/2}} + 2\mu \int_0^R \frac{re^{-\sqrt{\mu}r}}{(1+\lambda^2 r^2)^{1/2}} dr \right) \\
&= 2\pi \left(-\frac{2}{\lambda R} + O\left(\frac{1}{\lambda^3} + \frac{\sqrt{\mu}e^{-R\sqrt{\mu}}}{\lambda}\right) + \frac{2\mu}{\lambda^2}\int_0^{\lambda R} \frac{te^{-\frac{\sqrt{\mu}}{\lambda}t}}{(1+t^2)^{1/2}}dt \right) \\
&= 2\pi \left(-\frac{2}{\lambda R} + \frac{2\mu}{\lambda^2}\left(\int_0^{\lambda R} e^{-\frac{\sqrt{\mu}}{\lambda}t} dt\right)\right) + O\left(\frac{\mu}{\lambda^2} + \frac{\sqrt{\mu}e^{-R\sqrt{\mu}}}{\lambda}\right) \\
&= -\frac{4\pi}{\lambda R} + \frac{4\pi\sqrt{\mu}}{\lambda} + O\left(\frac{\mu}{\lambda^2} + \frac{\sqrt{\mu}e^{-R\sqrt{\mu}}}{\lambda}\right)
\end{aligned}
$$

$$
\begin{aligned}
\int_{B_R^+} \frac{(1-e^{\sqrt{\mu}|x|} - \sqrt{\mu}|x|e^{-\sqrt{\mu}|x|})^2}{\lambda |x|^4}dx &= \frac{2\pi}{\lambda}\int_0^R \frac{(1-e^{-\sqrt{\mu}r} - \sqrt{\mu}re^{-\sqrt{\mu}r})^2}{r^2}dr \\
&= \frac{2\pi\sqrt{\mu}}{\lambda}\int_0^{R\sqrt{\mu}} \frac{(1-e^{-t}-te^{-t})^2}{t^2}dt \\
&= \frac{2\pi\sqrt{\mu}}{\lambda}\left(\int_0^{+\infty} \frac{(1-e^{-t}-te^{-t})^2}{t^2}dt - \int_{R\sqrt{\mu}}^{+\infty}\left(\frac{1}{t^2} + O(\frac{e^{-t}}{t})\right)dt\right) \\
&= \frac{\pi\sqrt{\mu}}{\lambda} - \frac{2\pi}{\lambda R} + O\left(\frac{e^{-R\sqrt{\mu}}}{\lambda}\right) .
\end{aligned}
$$

Therefore

$$
\int_{B_R^+} |\nabla V|^2 = \frac{3\pi^2}{8} - 3\pi\frac{\sqrt{\mu}}{\lambda} + O\left(\frac{\mu}{\lambda^2} + \frac{\sqrt{\mu}e^{-R\sqrt{\mu}}}{\lambda}\right) . \tag{C.32}
$$

To estimate the contribution of the subdomains ω' and ω'' to the result, we consider the following expansions. Let $S > 0$ such that

$$
\lambda S \to +\infty \ , \ S\sqrt{\mu} \to 0 \quad \text{as } \lambda, \mu \to \infty \ , \ \frac{\sqrt{\mu}}{\lambda} \to 0 \, . \tag{C.33}
$$

According to (C.29), we have

$$(C.34)\qquad |\nabla V|^2 = \frac{\lambda^5|x|^2}{(1+\lambda^2|x|^2)^3} + O\left(\frac{\lambda^2\mu|x|}{(1+\lambda^2|x|^2)^{3/2}} + \frac{\mu^2}{\lambda}\right) \quad \text{for } |x| < 2S$$

just writing $e^{-\sqrt{\mu}|x|} = 1 - \sqrt{\mu}|x| + O(\mu|x|^2)$, and

$$(C.35)\qquad |\nabla V|^2 = \frac{(1+2\sqrt{\mu}|x| + \mu|x|^2)e^{-2\sqrt{\mu}|x|}}{\lambda|x|^4} + O\left(\frac{1}{\lambda^3|x|^6}\right) \quad \text{for } |x| > S$$

writing $(1+\lambda^2|x|^2)^{-\alpha} = \frac{1}{\lambda^{2\alpha}|x|^{2\alpha}}(1+O(\frac{1}{\lambda^2|x|^2}))$ for $|x| > S$. Note that, for μ large enough, $x \in (\omega' \cup \omega'')$, $|x'| \leq S$, implies $|x| < 2S$ because of (C.6) and (C.33). In view of (C.34) we compute, using (C.6) and (C.27)

$$\begin{aligned}\int_{\substack{\omega'\\|x'|\leq S}} \left(\tfrac{\lambda^2\mu|x|}{(1+\lambda^2|x|^2)^{3/2}} + \tfrac{\mu}{\lambda^2}\right) dx &= O\left(\int_0^S \left(\tfrac{\lambda^2\mu r}{(1+\lambda^2 r^2)^{3/2}} + \tfrac{\mu}{\lambda^2}\right) r^3 dr\right)\\ &= O\left(\tfrac{\mu}{\lambda^2}\int_0^{\lambda S} \tfrac{r^4 dr}{(1+r^2)^{3/2}} + \tfrac{\mu}{\lambda^2}S^4\right)\end{aligned}$$

whence, taking account of (C.33)

$$(C.36)\qquad \int_{\substack{\omega'\\|x'|\leq S}} \left(\frac{\lambda^2\mu|x|}{(1+\lambda^2|x|^2)^{3/2}} + \frac{\mu}{\lambda^2}\right) dx = O(\frac{\mu S^2}{\lambda})$$

and the same holds for ω'' instead of ω'. On the other hand, we have

$$\int_{\substack{\omega'\\|x'|\leq S}} \frac{\lambda^5|x|^2}{(1+\lambda^2|x|^2)^3}dx = \lambda^3 \int_{\substack{\omega'\\|x'|\leq S}} \left(\frac{1}{(1+\lambda^2|x|^2)^2} - \frac{1}{(1+\lambda^2|x|^2)^3}\right) dx\ .$$

The same holds for ω'' instead of ω', so that

$$\begin{aligned}&\int_{\substack{\omega'\\|x'|\leq S}} \frac{\lambda^5|x|^2}{(1+\lambda^2|x|^2)^3}dx - \int_{\substack{\omega''\\|x'|\leq S}} \frac{\lambda^5|x|^2}{(1+\lambda^2|x|^2)^3}\\ &= \lambda^3 \int_{|x'|\leq S} \left(\int_0^{f(x')} \left(\frac{1}{(1+\lambda^2|x|^2)^2} - \frac{1}{(1+\lambda^2|x|^2)^3}\right) dx_3\right) dx'\\ &= \lambda^2 \int_{|x'|\leq S} \left(\frac{1}{(1+\lambda^2|x'|^2)^{3/2}} \left(\int_0^{\frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}}} \frac{dz}{(1+z^2)^2}\right)\right.\\ &\qquad \left. - \frac{1}{(1+\lambda^2|x'|^2)^{5/2}} \left(\int_0^{\frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}}} \frac{dz}{(1+z^2)^3}\right)\right) dx'\\ &= \lambda^3 \sum_{i=1,2} f_i \int_{|x'|\leq S} \left(\frac{1}{(1+\lambda^2|x'|^2)^2} - \frac{1}{(1+\lambda^2|x'|^2)^3}\right) x_i^2 dx'\\ &\qquad + O\left(\lambda^3 \int_{|x'|\leq S} \frac{|x'|^3}{(1+\lambda^2|x'|^2)^2}dx'\right)\end{aligned}$$

using (C.6) and (C.21). This yields because of (C.7)

$$\int_{\substack{\omega' \\ |x'|\leq S}} \frac{\lambda^5|x|^2}{(1+\lambda^2|x|^2)^3}dx - \int_{\substack{\omega'' \\ |x'|\leq S}} \frac{\lambda^5|x|^2}{(1+\lambda^2|x|^2)^3}$$

$$= \frac{\pi H(0)}{\lambda}\int_0^{\lambda S}\left(\frac{1}{(1+r^2)^2}-\frac{1}{(1+r^2)^3}\right)r^3dr + O\left(\frac{1}{\lambda^2}\int_0^{\lambda S}\frac{r^4dr}{(1+r^2)^2}\right)$$

$$= \frac{\pi H(0)}{\lambda}\left(\int_1^{\lambda S}\frac{dr}{r}+\int_0^1\frac{r^3dr}{(1+r^2)^2}+\int_1^{+\infty}\left(\frac{r^3}{(1+r^2)^2}-\frac{1}{r}\right)dr-\int_0^{+\infty}\frac{r^3dr}{(1+r^2)^3}\right)$$

$$+O(\frac{1}{\lambda^3S^2}+\frac{S}{\lambda})$$

and finally, taking account of (C.34) (C.36) and (C.33), we obtain

$$\text{(C.37)}\qquad \int_{\substack{\omega' \\ |x'|\leq S}} |\nabla V|^2 - \int_{\substack{\omega'' \\ |x'|\leq S}} |\nabla V|^2 = \frac{\pi H(0)}{\lambda}\left(\mathrm{Log}\lambda S - \frac{3}{4}\right) + O\left(\frac{1}{\lambda^3S^2}+\frac{S}{\lambda}\right).$$

The last terms that we have to estimate are the integrals of $|\nabla V|^2$ for $x \in \omega'$ or $x \in \omega''$, $|x'| > S$. For $|x'| \geq a$, (C.35) shows that $|\nabla V|^2 = O\left(\frac{\mu e^{-2\sqrt{\mu}|x|}}{\lambda|x|^2}+\frac{1}{\lambda^3|x|^6}\right)$, hence

$$\text{(C.38)}\qquad \begin{aligned} &\int_{\substack{\omega' \\ |x'|> S}} |\nabla V|^2 - \int_{\substack{\omega'' \\ |x'|> S}} |\nabla V|^2 \\ &= \int_{\substack{\omega' \\ S<|x'|<a}} |\nabla V|^2 - \int_{\substack{\omega'' \\ S<|x'|<a}} |\nabla V|^2 + O\left(\frac{1}{\lambda^3}+\frac{\sqrt{\mu}e^{-2a\sqrt{\mu}}}{\lambda}\right).\end{aligned}$$

In view of (C.35) we compute, using (C.6)

$$\text{(C.39)}\qquad \int_{\substack{\omega' \\ S<|x'|<a}} \frac{dx}{\lambda^3|x|^6} \leq \frac{1}{\lambda^3}\int_{S<|x'|<a}\frac{dx'}{|x'|^4} = O\left(\frac{1}{\lambda^3S^2}\right)$$

and the same holds with ω'' instead of ω'. On the other hand we have, using (C.27), (C.6),

(C.7)

$$\int_{\substack{\omega' \\ S<|x'|<a}} \frac{(1+2\sqrt{\mu}|x|+\mu|x|^2)e^{-2\sqrt{\mu}|x|}}{\lambda|x|^4}dx - \int_{\substack{\omega'' \\ S<|x'|<a}} \frac{(1+2\sqrt{\mu}|x|+\mu|x|^2)e^{-2\sqrt{\mu}|x|}}{\lambda|x|^4}dx$$

$$= \frac{1}{\lambda}\int_{S<|x'|<a}\left(\int_0^{f(x')} \frac{(1+2\sqrt{\mu}|x'|+\mu|x'|^2+O(|x'|^2+\mu^{3/2}|x'|^5))e^{-2\sqrt{\mu}|x'|}}{|x'|^4}dx_3\right)dx'$$

$$= \frac{1}{\lambda}\sum_{i=1,2} f_i \int_{S<|x'|<a} \frac{(1+2\sqrt{\mu}|x'|+\mu|x'|^2)e^{-2\sqrt{\mu}|x'|}}{|x'|^4}x_i^2 dx'$$

$$+O\left(\frac{1}{\lambda}\int_{S<|x'|<a}\left(\frac{1}{|x'|}+\sqrt{\mu}+\mu|x'|+\mu^{3/2}|x'|^3\right)e^{-2\sqrt{\mu}|x'|}dx'\right)$$

$$= \frac{\pi H(0)}{\lambda}\int_{2\sqrt{\mu}S}^{2\sqrt{\mu}a}\left(\frac{1}{r}+1+\frac{r}{4}\right)e^{-r}dr + O\left(\frac{1}{\lambda\sqrt{\mu}}\int_{2\sqrt{\mu}S}^{2\sqrt{\mu}a}\left(1+r+r^2+\frac{r^3}{\sqrt{\mu}}\right)e^{-r}dr\right).$$

We have

$$\int_{2\sqrt{\mu}S}^{2\sqrt{\mu}a}\frac{e^{-r}}{r}dr = \int_{2\sqrt{\mu}S}^{1}\frac{dr}{r} + \int_{2\sqrt{\mu}S}^{1}\frac{e^{-r}-1}{r}dr + \int_1^{+\infty}\frac{e^{-r}}{r}dr - \int_{2\sqrt{\mu}a}^{+\infty}\frac{e^{-r}}{r}dr$$

$$= -\mathrm{Log}2\sqrt{\mu}S + \int_0^1 \frac{e^{-r}-1}{r}dr + \int_1^{+\infty}\frac{e^{-r}}{r}dr + O\left(\sqrt{\mu}S + \frac{e^{-2a\sqrt{\mu}}}{\sqrt{\mu}}\right)$$

$$= -\mathrm{Log}2\sqrt{\mu}S - \gamma + O\left(\sqrt{\mu}S + \frac{e^{-2a\sqrt{\mu}}}{\sqrt{\mu}}\right)$$

since

$$\int_0^1 \frac{e^{-r}-1}{r}dr + \int_1^{+\infty}\frac{e^{-r}}{r}dr = \int_0^{+\infty} e^{-r}\mathrm{Log}r\,dr = \Gamma'(1) = -\gamma\ ,$$

$$\int_{2\sqrt{\mu}S}^{2\sqrt{\mu}a}\left(1+\frac{r}{4}\right)e^{-r}dr = \frac{5}{4} + O\left(\sqrt{\mu}S + \sqrt{\mu}e^{-2a\sqrt{\mu}}\right)$$

and

$$\int_{2\sqrt{\mu}S}^{2\sqrt{\mu}a}(1+r+r^2+\frac{r^3}{\sqrt{\mu}})e^{-r}dr = O(1)\ .$$

It follows from these results and (C.38) (C.35) (C.39) that

$$\int_{\substack{\omega' \\ S<|x'|<a}}|\nabla V|^2 - \int_{\substack{\omega'' \\ S<|x'|<a}}|\nabla V|^2 = \frac{\pi H(0)}{\lambda}\left(-\mathrm{Log}2\sqrt{\mu}S - \gamma + \frac{5}{4}\right) + O\left(\frac{1}{\lambda\sqrt{\mu}} + \frac{\sqrt{\mu}S}{\lambda} + \frac{1}{\lambda^3 S^2}\right) \tag{C.40}$$

Together with (C.37), we find, choosing $S = \lambda^{-2/3}\mu^{-1/6}$ so that (C.33) is satisfied

$$\int_{\omega'} |\nabla V|^2 - \int_{\omega''} |\nabla V|^2 = \frac{\pi H(0)}{\lambda}\left(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} - \gamma + \frac{1}{2}\right) + O(\frac{1}{\lambda\sqrt{\mu}}) \tag{C.41}$$

and, considering (C.15) with $W = |\nabla V|^2$, (C.31) and (C.32), (C.1) is proved.

Remarks

1. Proceeding in the same way, we find

$$\int_{\Omega} |\nabla U_{\lambda,y}|^2 = \frac{3\pi^2}{8} - \pi H(y)\frac{\mathrm{Log}\lambda}{\lambda} + O(\frac{1}{\lambda}) \tag{C.42}$$

$$\int_{\Omega} |\nabla \varphi_{\lambda,y,\mu}|^2 = \pi\frac{\sqrt{\mu}}{\lambda} - \pi H(y)\frac{\mathrm{Log}\sqrt{\mu}}{\lambda} + O(\frac{1}{\lambda})\ . \tag{C.43}$$

Estimating further terms in these expansions would involve the whole shape of Ω, whereas (C.1) only involves the shape of Ω in a neighbourhood of the point $y \in \partial\Omega$ at which $V_{\lambda,y,\mu}$ concentrates as λ, μ go to infinity, because of the stronger decreasing of $V_{\lambda,\mu,\mu}$ away from this point.

2. The estimates that we obtained may be derivated with respect to λ. Indeed, using the same arguments as previously, we would get

$$\int_{\Omega} \nabla V_{\lambda,y,\mu}.\nabla\frac{\partial V_{\lambda,y,\mu}}{\partial\lambda} = \frac{3\pi}{2}\frac{\sqrt{\mu}}{\lambda^2} + \frac{\pi}{2}\frac{H(y)}{\lambda^2}(\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} + \gamma - \frac{1}{2}) + O(\frac{1}{\lambda^2\sqrt{\mu}}) \tag{C.44}$$

$$\int_{\Omega} V_{\lambda,y,\mu}\frac{\partial V_{\lambda,y,\mu}}{\partial\lambda} = -\frac{\pi}{2\lambda^2\sqrt{\mu}} + \frac{\pi}{4}\frac{H(y)}{\lambda^2\mu} + O(\frac{1}{\lambda^2\mu^{3/2}}) \tag{C.45}$$

$$\int_{\Omega} V^5_{\lambda,y,\mu}\frac{\partial V_{\lambda,y,\mu}}{\partial\lambda} = \frac{2\pi}{3}\frac{\sqrt{\mu}}{\lambda^2} + \frac{\pi}{24}\frac{H(y)}{\lambda^2} + O(\frac{\sqrt{\mu}}{\lambda^2}) \tag{C.46}$$

and

(C.47)
$$\frac{\partial K_\mu}{\partial\lambda}(\lambda,y,0) = 2\pi^{1/3}\left(-2\frac{\sqrt{\mu}}{\lambda^2} + \frac{H(y)}{\lambda^2}\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} - \gamma + \frac{1}{2}\right) + O\left(\frac{1}{\lambda^2\sqrt{\mu}} + \frac{\mu}{\lambda^3} + \frac{\sqrt{\mu}}{\lambda^3}\mathrm{Log}\frac{\lambda}{\sqrt{\mu}}\right)\ .$$

In the same way

(C.48)
$$\frac{\partial^2 K_\mu}{\partial\lambda^2}(\lambda,y,0) = 4\pi^{1/3}\left(2\frac{\sqrt{\mu}}{\lambda^3} - \frac{H(y)}{\lambda^3}\mathrm{Log}\frac{\lambda}{2\sqrt{\mu}} - \gamma - \frac{1}{2}\right) + O\left(\frac{1}{\lambda^3\sqrt{\mu}} + \frac{\mu}{\lambda^4} + \frac{\sqrt{\mu}}{\lambda^4}\mathrm{Log}\frac{\lambda}{\sqrt{\mu}}\right)\ .$$

We end this appendix with some easy estimates which occur in the arguments of Section 3. Still assuming that μ goes to infinity and $\frac{\sqrt{\mu}}{\lambda}$ goes to zero, we prove :

$$\left(\int_\Omega U_{\lambda,y}^{24/5}\varphi_{\lambda,y,\mu}^{6/5}\right)^{5/6} = O\left(\frac{\sqrt{\mu}}{\lambda}\right) \tag{C.49}$$

$$\left(\int_\Omega U_{\lambda,y}^{9/2}\varphi_{\lambda,y,\mu}^{3/2}\right)^{2/3} = O\left(\frac{\sqrt{\mu}}{\lambda}\right) \tag{C.50}$$

$$\left(\int_\Omega \left|U_{\lambda,y} - \frac{1}{\lambda^{1/2}|x-y|}\right|^{6/5}\right)^{5/6} = O\left(\frac{1}{\lambda^2}\right) \tag{C.51}$$

$$\left(\int_\Omega \left|\frac{\partial V}{\partial \nu}\right|^{4/3}\right)^{3/4} = O\left(\frac{1}{\lambda^{1/2}\mu^{1/4}}\right) \tag{C.52}$$

Proof. Again, we may assume that $y = 0$, and that (C.5) (C.6) (C.8) hold. According to the definitions (1.9) and (2.1) of $U = U_{\lambda,0}$ and $\varphi = \varphi_{\lambda,0,\mu}$, we have

$$\begin{aligned}\int_\Omega U^{24/5}\varphi^{6/5} &\leq 4\pi\int_0^{R'} \frac{\lambda^{12/5}}{(1+\lambda^2r^2)^{12/5}}\frac{(1-e^{-\sqrt{\mu}r})^{6/5}}{\lambda^{3/5}r^{6/5}}r^2dr \\ &\leq 4\pi\lambda^{9/5}\mu^{3/5}\int_0^{R'}\frac{r^2dr}{(1+\lambda^2r^2)^{12/5}}\end{aligned}$$

using (C.8) and the inequalities $0 \leq 1 - e^{-\sqrt{\mu}r} \leq \sqrt{\mu}r$. Therefore

$$\int_\Omega U^{24/5}\varphi^{6/5} \leq 4\pi\frac{\mu^{3/5}}{\lambda^{6/5}}\int_0^{\lambda R'}\frac{r^2dr}{(1+r^2)^{12/5}}$$

and (C.49) follows. The same kind of computations leads to (C.50). Concerning (C.51), we write

$$\begin{aligned}\int_\Omega \left|U - \tfrac{1}{\lambda^{1/2}|x|}\right|^{6/5} &\leq 4\pi\int_0^{R'}\left(\tfrac{1}{\lambda^{1/2}r} - \tfrac{\lambda^{1/2}}{(1+\lambda^2r^2)^{1/2}}\right)^{6/5} r^2dr \\ &\leq \tfrac{4\pi}{\lambda^{12/5}}\int_0^{\lambda R'}\left(\tfrac{1}{r} - \tfrac{1}{(1+r^2)^{1/2}}\right)^{6/5} r^2dr\end{aligned}$$

hence the result since the integral is bounded as λ goes to infinity. It only remains to prove (C.52). According to (C.28), we see that

$$\frac{\partial V}{\partial x_i} = O\left(\frac{1}{\lambda^{5/2}} + \frac{e^{-\frac{R}{2}\sqrt{\mu}}}{\lambda^{1/2}}\right) \text{ uniformly on } \Omega\cap B_{R/2}^c, i = 1, 2, 3$$

whence

$$\text{(C.53)} \qquad \int_{\partial\Omega\cap B^c_{R/2}} |\frac{\partial V}{\partial \nu}|^{4/3} = O\left(\frac{1}{\lambda^{10/3}} + \frac{e^{-\frac{2}{3}R\sqrt{\mu}}}{\lambda^{2/3}}\right) .$$

On $\partial\Omega \cap B_R$, using (C.6) and (C.28), we have

$$\begin{aligned} \frac{\partial V}{\partial \nu} &= \frac{1}{(1+f^2(x'))^{1/2}}\left(\sum_{i=1,2}\frac{\partial f}{\partial x_i}\frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial x_3}\right) \\ &= O\left(\left|-\frac{\lambda^{5/2}}{(1+\lambda^2|x|^2)^{3/2}} + \frac{1-e^{-\sqrt{\mu}|x|}-\sqrt{\mu}|x|e^{-\sqrt{\mu}|x|}}{\lambda^{1/2}|x|^3}\right| |x'|^2\right) . \end{aligned}$$

As a consequence

$$\text{(C.54)} \qquad \frac{\partial V}{\partial \nu} = O\left(\lambda^{5/2}|x'|^2 + \frac{\mu|x'|}{\lambda^{1/2}}\right) \quad \text{for } x \in \partial\Omega \cap B_R, |x'| \le \frac{1}{\lambda}$$

and, using (C.27)

$$\text{(C.55)} \qquad \frac{\partial V}{\partial \nu} = O\left(\frac{e^{-\sqrt{\mu}|x'|}(1+\sqrt{\mu}|x'|)}{\lambda^{1/2}|x'|} + \frac{1}{\lambda^{5/2}|x'|^3}\right) \quad \text{for } x \in \partial\Omega \cap B_R, \frac{1}{\lambda} \le |x'| < \frac{R}{2}$$

assuming that for $|x'| < \frac{R}{2}$, $|x| = (|x'|^2 + f^2(x'))^{1/2} < R$, which is true provided that $R > 0$ is chosen small enough, because of (C.7). Then

$$\begin{aligned} \int_{\partial\Omega\cap B_{R/2}} |\frac{\partial V}{\partial \nu}|^{4/3} d\sigma = O\Bigg(&\int_{|x'|\le 1/\lambda}\left(\lambda^{5/2}|x'|^2 + \frac{\mu|x'|}{\lambda^{1/2}}\right)^{4/3} dx' \\ &+ \int_{1/\lambda \le |x'| \le R/2}\left(\frac{e^{-\sqrt{\mu}|x'|}(1+\sqrt{\mu}|x'|)}{\lambda^{1/2}|x'|} + \frac{1}{\lambda^{5/2}|x'|^3}\right)^{4/3} dx'\Bigg) \end{aligned}$$

noticing that $|\frac{d\sigma}{dx'}| = (1+f'^2(x'))^{1/2} = O(1)$. On one hand

$$\begin{aligned} \int_{|x'|\le 1/\lambda}\left(\lambda^{5/2}|x'|^2 + \frac{\mu|x'|}{\lambda^{1/2}}\right)^{4/3} dx' &= O\left(\int_0^{1/\lambda}(\lambda^{10/3}r^{11/3} + \frac{\mu}{\lambda^{2/3}}r^{7/3})dr\right) \\ &= O(\frac{1}{\lambda^{4/3}}) . \end{aligned}$$

On the other hand

$$\int_{1/\lambda\le|x'|<\frac{R}{2}}\left(\frac{e^{-\sqrt{\mu}|x'|}(1+\sqrt{\mu}|x'|)}{\lambda^{1/2}|x'|}+\frac{1}{\lambda^{5/2}|x'|^3}\right)^{4/3}dx'$$

$$=O\left(\frac{1}{\lambda^{2/3}}\int_{1/\lambda}^{R/2}e^{-\frac{4}{3}\sqrt{\mu}r}(\frac{1}{r^{1/3}}+\mu^{2/3}r)dr+\frac{1}{\lambda^{10/3}}\int_{1/\lambda}^{R/2}\frac{dr}{r^3}\right)$$

$$=O\left(\frac{1}{\lambda^{2/3}\mu^{1/3}}\int_{4\sqrt{\mu}/3\lambda}^{2\sqrt{\mu}/3}e^{-t}(\frac{1}{t^{1/3}}+t)dt+\frac{1}{\lambda^{4/3}}\right)$$

$$=O\left(\frac{1}{\lambda^{2/3}\mu^{1/3}}\right)$$

so that, finally

$$\int_{\partial\Omega\cap B_{R/2}}|\frac{\partial V}{\partial\nu}|^{4/3}=O(\frac{1}{\lambda^{2/3}\mu^{1/3}})\ .$$

(C.53) shows that the same estimate holds integrating on whole $\partial\Omega$, hence the desired result.

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